Caputo delta weakly fractional difference equations

M. Fečkan[a,b], M. Pospíšil[a,b], M.-F. Danca[c,d], and J. R. Wang[e]

[a]Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University, Bratislava, Slovakia; [b]Mathematical Institute of Slovak Academy of Sciences, Bratislava, Slovakia; [c]Department of Mathematics and Computer Science, Avram Iancu University of Cluj-Napoca, Cluj-Napoca, Romania; [d]Romanian Institute of Science and Technology, Cluj-Napoca, Romania; [e]Department of Mathematics, Guizhou University, Guiyang, Guizhou, China

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ABSTRACT

In this paper, solutions of fractional difference equations with Caputo-type delta-based fractional difference operator of order $\mu \sim 1$ are compared with solutions of corresponding difference equations with usual first-order forward difference. To derive convergence results, Gronwall type inequalities are proved for suitable fractional sum inequalities of general noninteger order. An illustrative example is also given.

KEYWORDS

Weakly fractional difference equation; Caputo fractional difference; comparison; discrete Mittag-Leffler function; fractional Gronwall inequality

1. Introduction

Beginnings of discrete fractional calculus are due to Miller and Ross [10]. Later Atici and Eloe proved further properties of the fractional sum operator in [3]. Caputo like fractional difference was established in [1] (cf. also [2]).

Recently in [8], convergence was investigated of solutions of fractional differential equations with Caputo fractional derivative of order close to 1 to solutions of corresponding differential equations of the first order as the fractional order tends to 1. It was shown that the corresponding integer-order equation substantially depends on the side of the one-sided limit. In this paper, we compare a solution of a fractional difference equation with Caputo like delta-based fractional difference of order μ close to 1 with a solution of a corresponding difference equation of order 1. So, the present paper can be considered as a discrete counterpart to [8].

In the following section, we conclude preliminary results needed for main sections. There are also proved Gronwall type inequalities for fractional sum inequalities of any non-integer order. Sections 3 and 4 are devoted to cases $\mu \to 1^-$ and $\mu \to 1^+$, respectively. A simple example is given at the end of Section 4 to illustrate the convergence results.

Here and after, $\mathbb{N}_a$, $a \in \mathbb{R}$ denotes the shifted set of positive integers, i.e., $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$. We shortly denote $\mathbb{N} := \mathbb{N}_1$. We also use $\mathbb{Z}_a^b := \{a, a+1, \dots, b\}$ for $b - a \in \mathbb{N}_0$, $\mathbb{Z}_a^b = \emptyset$ if $a > b$, and $\mathbb{R}_+ := [0, \infty)$.

Throughout the paper, we assume the property of empty sum and empty product, i.e.,

$$\sum_{k=a}^{b} f(k) = 0, \qquad \prod_{k=a}^{b} f(k) = 1$$

if $a > b$.

2. Preliminaries

First we recall some definitions from the theory of fractional difference calculus. Basic definitions are due to [1, 10]. For properties of fractional difference operator see also [3, 4].

Definition 2.1. Let $\nu \in \mathbb{R}$. Factorial function is defined as

$$t^{(\nu)} = \begin{cases} 0, & t+1-\nu \in \{\dots, -2, -1, 0\}, \\ \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}, & \text{otherwise} \end{cases}$$

where Γ is the Euler gamma function.

Definition 2.2. Let $a \in \mathbb{R}$, $\nu > 0$. The ν-th fractional sum of function f defined on $\mathbb{N}_a$ is given by

$$\Delta^{-\nu} f(k) := (\Delta_a^{-\nu} f)(k) = \frac{1}{\Gamma(\nu)} \sum_{j=a}^{k-\nu} (k - \sigma(j))^{(\nu-1)} f(j)$$

for any $k \in \mathbb{N}_{a+\nu}$.

Definition 2.3. Let $a \in \mathbb{R}$, $\mu > 0$, $m - 1 < \mu < m$ for some $m \in \mathbb{N}$, $\nu := m - \mu$ and function f be defined on $\mathbb{N}_a$. The μ-th fractional Caputo like difference of f with the lower limit at a is defined as

$$\Delta_*^{\mu} f(k) := ({}^{C}\Delta_a^{\mu} f)(k) = (\Delta^{-\nu}(\Delta^m f))(k) = \frac{1}{\Gamma(\nu)} \sum_{j=a}^{k-\nu} (k - \sigma(j))^{(\nu-1)} (\Delta^m f)(j)$$

for any $k \in \mathbb{N}_{a+\nu}$. Here Δ^m is the m-th forward difference operator,

$$(\Delta^m f)(k) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(k+j).$$

For the simplicity, in the rest of the paper, we consider the fractional difference with the lower limit at 0.

We shall need the following estimations of a ratio of gamma functions (see also [11]).

Lemma 2.4 (see [12]). *For any* $0 < s < 1$ *and* $x > 0$,

$$x^{1-s} \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq (x+s)^{1-s}.$$

Lemma 2.5 (see [9]). *For any* $0 < s < 1$ *and* $x > 0$,

$$\frac{\Gamma(x+s)}{\Gamma(x+1)} < \left(x + \frac{s}{2}\right)^{s-1}.$$

Next we present two Gronwall type inequalities for fractional sums.

Lemma 2.6. *Let* $0 < \mu < 1$, $z, a\colon \mathbb{Z}_0^K \to \mathbb{R}_+$ *for some* $K \in \mathbb{N} \cup \{\infty\}$ *and* a *be nondecreasing. If there is* $L > 0$ *such that*

$$z(k) \leq a(k) + \frac{L}{\Gamma(\mu)} \sum_{j=0}^{k-1} (k - \sigma(j+1-\mu))^{(\mu-1)} z(j), \qquad k \in \mathbb{Z}_0^K$$

then

$$z(k) \leq a(k) E_\mu \left(\frac{L k^\mu}{\mu + \mu^2} \right), \qquad k \in \mathbb{Z}_0^K,$$

where $E_\mu(w) = \sum_{j=0}^\infty \frac{w^k}{\Gamma(j\mu+1)}$ *is the Mittag-Leffler function.*

Proof. Let us extend functions z and a to $[0, T)$, $T := K + 1$ by $z(t) = z(\lfloor t \rfloor)$ and $a(t) = a(\lfloor t \rfloor)$ where $\lfloor \cdot \rfloor$ is the floor function giving the greatest lower integer. Let us fix arbitrary $t \in [0, T)$. Then

$$\begin{aligned} z(t) &\leq a(t) + \frac{L}{\Gamma(\mu)} \sum_{j=0}^{k-1} (k - \sigma(j+1-\mu))^{(\mu-1)} z(j) \\ &= a(t) + \frac{L}{\Gamma(\mu)} \sum_{j=0}^{k-1} \int_j^{j+1} (k - \sigma(j+1-\mu))^{(\mu-1)} z(s) ds \end{aligned}$$

with $k = \lfloor t \rfloor$. Now using the definition of the factorial function and Lemma 2.4, we derive for $j \in \mathbb{Z}_0^{k-1}$,

$$\begin{aligned} (k - \sigma(j+1-\mu))^{(\mu-1)} &= \frac{\Gamma(k-j-1+\mu)}{\Gamma(k-j)} \\ &= \frac{k-j}{k-j-1+\mu} \frac{k-j+1}{k-j+\mu} \frac{\Gamma(k-j+1+\mu)}{\Gamma(k-j+2)} \\ &= \left(1 + \frac{1-\mu}{k-j-1+\mu}\right) \left(1 + \frac{1-\mu}{k-j+\mu}\right) \frac{\Gamma(k-j+1+\mu)}{\Gamma(k-j+2)} \\ &\leq \left(1 + \frac{1-\mu}{\mu}\right) \left(1 + \frac{1-\mu}{1+\mu}\right) (k-j+1)^{\mu-1} \\ &= \frac{2}{\mu(1+\mu)} (k-j+1)^{\mu-1}. \end{aligned}$$

Hence,

$$
\begin{aligned}
z(t) &\leq a(t) + \frac{2L}{\Gamma(2+\mu)} \sum_{j=0}^{k-1} \int_j^{j+1} (k-j+1)^{\mu-1} z(s) ds \\
&\leq a(t) + \frac{2L}{\Gamma(2+\mu)} \sum_{j=0}^{k-1} \int_j^{j+1} (k-s+1)^{\mu-1} z(s) ds \\
&= a(t) + \frac{2L}{\Gamma(2+\mu)} \int_0^k (k-s+1)^{\mu-1} z(s) ds \\
&\leq a(t) + \frac{2L}{\Gamma(2+\mu)} \int_0^t (k-s+1)^{\mu-1} z(s) ds \\
&\leq a(t) + \frac{2L}{\Gamma(2+\mu)} \int_0^t (t-s)^{\mu-1} z(s) ds.
\end{aligned}
$$

By the Henry–Gronwall inequality [13, Corollary 2] we get

$$
z(t) \leq a(t) E_\mu \left(\frac{2Lt^\mu}{\mu+\mu^2} \right)
$$

for any $t \in [0, T)$. The statement is obtained by setting $t = k$. □

Lemma 2.7. *Let $m-1 < \mu < m$ for some $m \in \mathbb{N}_2$, $z, a \colon \mathbb{Z}_0^K \to \mathbb{R}_+$ for some $K \in \mathbb{N}_m \cup \{\infty\}$ and a be nondecreasing. If there is $L > 0$ such that*

$$
z(k) \leq a(k) + \frac{L}{\Gamma(\mu)} \sum_{j=0}^{k-m} (k - \sigma(j+m-\mu))^{(\mu-1)} z(j), \qquad k \in \mathbb{Z}_0^K
$$

then

$$
z(k) \leq a(k) E_\mu(Lk^\mu), \qquad k \in \mathbb{Z}_0^K.
$$

Proof. Let $\mu \in (m-1, m)$ be fixed for some $m \in \mathbb{N}_2$. The statement is proved as the previous one using the following estimations:

$$
\begin{aligned}
(k - \sigma(j+m-\mu))^{(\mu-1)} &= \frac{\Gamma(k-j-m+\mu)}{\Gamma(k-j-m+1)} \\
&= \frac{\Gamma(k-j-2m+\mu+2)}{\Gamma(k-j-m+1)} \prod_{l=1}^{m-2} (k-j-m+\mu-l) \\
&\leq (k-j-m+1)^{\mu-m+1} \prod_{l=1}^{m-2} (k-j-m+\mu-l) \\
&\leq (k-j-m+1)^{\mu-m+1} \prod_{l=1}^{m-2} (k-j-l) \leq (k-j-1)^{\mu-1}
\end{aligned}
$$

for each fixed $j \in \mathbb{Z}_0^{k-m}$, $k \in \mathbb{Z}_0^K$, and

$$\sum_{j=0}^{k-m} (k-\sigma(j+m-\mu))^{(\mu-1)} z(j) = \sum_{j=0}^{k-m} \int_j^{j+1} (k-\sigma(j+m-\mu))^{(\mu-1)} z(s) ds$$
$$\leq \sum_{j=0}^{k-m} \int_j^{j+1} (k-j-1)^{\mu-1} z(s) ds \leq \sum_{j=0}^{k-m} \int_j^{j+1} (k-s)^{\mu-1} z(s) ds$$
$$= \int_0^{k-m+1} (k-s)^{\mu-1} z(s) ds \leq \int_0^{k-m+1} (t-s)^{\mu-1} z(s) ds \leq \int_0^t (t-s)^{\mu-1} z(s) ds$$

for any $t \in [0,T)$, $T = K+1$, where $z(s) = z(\lfloor s \rfloor)$, $k = \lfloor t \rfloor$. □

Usually, a fractional sum equation equivalent to an initial value problem for a fractional difference equation of order $\mu > 0$ is stated for $\mu \in (0,1)$ (see e.g. [5, Lemma 2.4]). For the convenience of the reader, here we state the result for $\mu \in (1,2)$.

Lemma 2.8. *Let $\mu \in (1,2)$, $x_0, x_1 \in \mathbb{R}^n$ and $f\colon \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ be a given function. Function $x\colon \mathbb{N}_0 \to \mathbb{R}^n$ is a solution of*

$$\begin{aligned} \Delta_*^\mu x(k) &= f(k+\mu-2, x(k+\mu-2)), \qquad k \in \mathbb{N}_{2-\mu}, \\ x(0) &= x_0, \\ \Delta x(0) &= x_1 \end{aligned} \tag{1}$$

if and only if it satisfies

$$x(k) = x_0 + kx_1 + \frac{1}{\Gamma(\mu)} \sum_{j=0}^{k-2} (k-\sigma(j+2-\mu))^{(\mu-1)} f(j, x(j)) \tag{2}$$

for each $k \in \mathbb{N}_0$ (assuming the empty sum property).

Proof. If x is a solution of (1), [1, Theorem 8] yields that x fulfills (2). Note that this can be obtained directly by applying the operator $\Delta_{2-\mu}^{-\mu}$ to the fractional difference equation and then using the initial conditions.

Conversely, if x satisfies (2), then $x(0) = x_0$, $\Delta x(0) = x(1) - x(0) = x_1$ and

$$x(k) = x_0 + kx_1 + \frac{1}{\Gamma(\mu)} \sum_{j=2-\mu}^{k-\mu} (k-\sigma(j))^{(\mu-1)} f(j+\mu-2, x(j+\mu-2))$$

for each $k \in \mathbb{N}_2$. On the other side, by [1, Theorem 8],

$$x(k) = x(0) + k\Delta x(0) + \frac{1}{\Gamma(\mu)} \sum_{j=2-\mu}^{k-\mu} (k-\sigma(j))^{(\mu-1)} \Delta_*^\mu x(j)$$

for each $k \in \mathbb{N}_2$. Comparing these two equations, we get

$$\frac{1}{\Gamma(\mu)} \sum_{j=2-\mu}^{k-\mu} (k-\sigma(j))^{(\mu-1)} [\Delta_*^\mu x(j) - f(j+\mu-2, x(j+\mu-2))] = 0$$

for each $k \in \mathbb{N}_2$. Now, subsequently letting $k = 2, 3, \dots$ we derive $\Delta_*^\mu x(k) = f(k+\mu-2, x(k+\mu-2))$ for each $k \in \mathbb{N}_{2-\mu}$. □

Next, we recall estimations of the Mittag-Leffler function for various values of the parameter μ.

Lemma 2.9 (see [6, Lemma 2]). *For all $t \in \mathbb{R}_+$, $\mu \in (0,1)$, and $\kappa > 0$, it holds*

$$1 \leq E_\mu(\kappa t^\mu) \leq \frac{\mathrm{e}^{\kappa^{\frac{1}{\mu}} t}}{\mu}.$$

Lemma 2.10 (see [8, Lemma 3.1]). *For all $t \in \mathbb{R}_+$, $\mu \in (1, 4/3)$, and $\kappa > 0$, it holds*

$$1 \leq E_\mu(\kappa t^\mu) \leq \frac{\mathrm{e}^{\kappa^{\frac{1}{\mu}} t}}{\mu} + \frac{4\sqrt{3} \sin \frac{\pi\mu}{2}}{9\mu}.$$

3. The case $\mu \to 1^-$

Let us consider an initial value problem for fractional difference equation

$$\begin{aligned} \Delta_*^\mu x(k) &= f(k+\mu-1, x(k+\mu-1)), \qquad k \in \mathbb{N}_{1-\mu}, \\ x(0) &= x_0 \end{aligned} \tag{3}$$

where Δ_*^μ is the Caputo fractional difference of order $\mu \in (0,1)$ with the lower limit at zero and $f \colon \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function, along with a difference equation

$$\begin{aligned} \Delta y(k) &= f(k, y(k)), \qquad k \in \mathbb{N}_0, \\ y(0) &= y_0 \end{aligned} \tag{4}$$

for $x_0, y_0 \in \mathbb{R}^n$. We suppose

(H) There are nonnegative constants M and L such that $\|f(k,x)\| \leq M$ and $\|f(k,x) - f(k,y)\| \leq L\|x-y\|$ for each $k \in \mathbb{N}_0$ and all $x, y \in \mathbb{R}^n$, where $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

From [5, Lemma 2.4] we know that $x(k)$ is a solution of (3) if and only if it satisfies

$$\begin{aligned} x(k) &= x_0 + \frac{1}{\Gamma(\mu)} \sum_{j=1-\mu}^{k-\mu} (k-\sigma(j))^{(\mu-1)} f(j+\mu-1, x(j+\mu-1)) \\ &= x_0 + \frac{1}{\Gamma(\mu)} \sum_{j=0}^{k-1} (k-\sigma(j+1-\mu))^{(\mu-1)} f(j, x(j)) \end{aligned}$$

for each $k \in \mathbb{N}_0$ (assuming the empty sum property). Moreover, $y(k)$ solves (4) if and only if

$$y(k) = y_0 + \sum_{j=0}^{k-1} f(j, y(j)), \qquad k \in \mathbb{N}_0.$$

Hence, for each $k \in \mathbb{N}_0$,

$$\begin{aligned}
\|x(k) - y(k)\| &\leq \|x_0 - y_0\| + \left\| \sum_{j=0}^{k-1} \left(\frac{(k - \sigma(j+1-\mu))^{(\mu-1)}}{\Gamma(\mu)} f(j, x(j)) - f(j, y(j)) \right) \right\| \\
&\leq \|x_0 - y_0\| + \frac{1}{\Gamma(\mu)} \sum_{j=0}^{k-1} (k - \sigma(j+1-\mu))^{(\mu-1)} \|f(j, x(j)) - f(j, y(j))\| \\
&\quad + \sum_{j=0}^{k-1} \left| 1 - \frac{(k - \sigma(j+1-\mu))^{(\mu-1)}}{\Gamma(\mu)} \right| \|f(j, y(j))\| \\
&\leq \|x_0 - y_0\| + \frac{L}{\Gamma(\mu)} \sum_{j=0}^{k-1} (k - \sigma(j+1-\mu))^{(\mu-1)} \|x(j) - y(j)\| \\
&\quad + M \sum_{j=0}^{k-1} \left| 1 - \frac{(k - \sigma(j+1-\mu))^{(\mu-1)}}{\Gamma(\mu)} \right| \\
&= \|x_0 - y_0\| + \frac{L}{\Gamma(\mu)} \sum_{j=0}^{k-1} (k - \sigma(j+1-\mu))^{(\mu-1)} \|x(j) - y(j)\| \\
&\quad + M \sum_{j=0}^{k-1} \left| 1 - \frac{(j - \sigma(-\mu))^{(\mu-1)}}{\Gamma(\mu)} \right|.
\end{aligned}$$

Applying Lemma 2.6 yields

$$\|x(k) - y(k)\| \leq \left(\|x_0 - y_0\| + M \sum_{j=0}^{k-1} \left| 1 - \frac{(j - \sigma(-\mu))^{(\mu-1)}}{\Gamma(\mu)} \right| \right) E_\mu(L_1 k^\mu)$$

for each $k \in \mathbb{N}_0$, where $L_1 = \frac{2L}{\mu + \mu^2}$. We continue with the case $x_0 = y_0$. Then we have

$$\|x(k) - y(k)\| \leq M\theta_\mu(k), \quad \theta_\mu(k) = E_\mu(L_1 k^\mu) \sum_{j=0}^{k-1} \left| 1 - \frac{(j - \sigma(-\mu))^{(\mu-1)}}{\Gamma(\mu)} \right| \tag{5}$$

for each $k \in \mathbb{N}_0$. Clearly, $\theta_\mu(0) = 0$ for any $\mu \in (0, 1)$. From now on, we consider $k \in \mathbb{N}$.

Let us investigate the sum in θ_μ: Clearly, the summand vanishes for $j = 0$. Moreover, for $j \in \mathbb{Z}_1^{k-1}$, $k \in \mathbb{N}_2$ we have

$$\frac{(j - \sigma(-\mu))^{(\mu-1)}}{\Gamma(\mu)} = \frac{\Gamma(j+\mu)}{\Gamma(\mu)\Gamma(j+1)} = \prod_{l=1}^{j} \frac{j + \mu - l}{j + 1 - l} < 1. \tag{6}$$

So, we can remove the absolute value and write

$$\sum_{j=0}^{k-1} \left| 1 - \frac{(j-\sigma(-\mu))^{(\mu-1)}}{\Gamma(\mu)} \right| = k - 1 - \frac{1}{\Gamma(\mu)} \sum_{j=1}^{k-1} \frac{\Gamma(j+\mu)}{\Gamma(j+1)}.$$

Next, we rewrite the sum as a telescoping series,

$$\begin{aligned} \sum_{j=1}^{k-1} \frac{\Gamma(j+\mu)}{\Gamma(j+1)} &= \frac{1}{\mu} \sum_{j=1}^{k-1} \left[\frac{\Gamma(j+\mu+1)}{\Gamma(j+1)} - \frac{\Gamma(j+\mu)}{\Gamma(j)} \right] \\ &= \frac{1}{\mu} \left[\frac{\Gamma(k+\mu)}{\Gamma(k)} - \frac{\Gamma(1+\mu)}{\Gamma(1)} \right] \end{aligned} \tag{7}$$

to get

$$\sum_{j=0}^{k-1} \left| 1 - \frac{(j-\sigma(-\mu))^{(\mu-1)}}{\Gamma(\mu)} \right| = k - \frac{\Gamma(k+\mu)}{\Gamma(1+\mu)\Gamma(k)}, \quad k \in \mathbb{N}_2. \tag{8}$$

One can easily verify that the latter identity holds for each $k \in \mathbb{N}$.

Theorem 3.1. *Under assumption* (H)*, the solution x of* (3) *uniformly converges on any set $\mathbb{Z}_0^K$, $K \in \mathbb{N}$ to the solution y of* (4) *if $\mu \to 1^-$ and $x_0 = y_0$.*

Proof. Note that the limit

$$\lim_{\mu \to 1^-} \left(k - \frac{\Gamma(k+\mu)}{\Gamma(1+\mu)\Gamma(k)} \right) = 0$$

is uniform with respect to $k \in \mathbb{Z}_1^K$, since

$$\begin{aligned} 0 \le k - \frac{\Gamma(k+\mu)}{\Gamma(1+\mu)\Gamma(k)} &= k \left(1 - \frac{\Gamma(k+\mu)}{\Gamma(1+\mu)\Gamma(k+1)} \right) \\ \le k \left(1 - \frac{(k+\mu)^{\mu-1}}{\Gamma(1+\mu)} \right) &\le K \left(1 - \frac{(K+\mu)^{\mu-1}}{\Gamma(1+\mu)} \right) \xrightarrow{\mu \to 1^-} 0 \end{aligned}$$

due to Lemma 2.4. Consequently, estimation of $E_\mu(L_1 k^\mu)$ given by Lemma 2.9 together with identity (8) proves the statement. □

Noting that

$$\theta_\mu(k) = \begin{cases} 0, & k = 0, \\ \left(k - \frac{\Gamma(k+\mu)}{\Gamma(1+\mu)\Gamma(k)} \right) E_\mu(L_1 k^\mu), & k \in \mathbb{N} \end{cases}$$

is increasing on $\mathbb{N}$ from 0 to ∞ (see (6) and (8)) together with Theorem 3.1 proves the next result.

Theorem 3.2. *Under assumption* (H)*, for any* $\varepsilon > 0$*,* $\mu \in (0,1)$ *there exists* $K \in \mathbb{N}$ *such that the solution* x *of* (3) *and* y *of* (4) *with* $x_0 = y_0$ *satisfy*

$$\|x(k) - y(k)\| \leq M\varepsilon, \qquad k \in \mathbb{Z}_0^K.$$

This K *is given as a largest integer such that*

$$\frac{\mathrm{e}^{L_1^{\frac{1}{\mu}} K}}{\mu} \left(K - \frac{\Gamma(K+\mu)}{\Gamma(1+\mu)\Gamma(K)} \right) \leq \varepsilon.$$

To provide a result not so strong as the latter one but more easy to apply, we state the following corollary.

Corollary 3.3. *Under* (H)*, for any* $\varepsilon > 0$*,* $\mu \in (0,1)$*, solutions* x *of* (3) *and* y *of* (4) *with* $x_0 = y_0$ *satisfy*

$$\|x(k) - y(k)\| \leq M\varepsilon, \qquad k \in \mathbb{Z}_0^K$$

with $K \in \mathbb{N}$ *the largest integer satisfying*

$$(1-\mu)\,\mathrm{e}^{L_1^{\frac{1}{\mu}} K}\, K \ln K \leq \varepsilon\mu.$$

In particular, for $\varepsilon = (1-\mu)^p$ *for any fixed* $p \in (0,1)$*,*

$$\|x(k) - y(k)\| \leq M(1-\mu)^p, \qquad k \in \mathbb{Z}_0^K, \tag{9}$$

where $K \in \mathbb{N}$ *is the largest integer satisfying*

$$\mathrm{e}^{L_1^{\frac{1}{\mu}} K}\, K \ln K \leq \frac{\mu}{(1-\mu)^{1-p}}. \tag{10}$$

Proof. Let $K \in \mathbb{N}_2$. Then the following inequality holds

$$\begin{aligned} 1 - \frac{\Gamma(K+\mu)}{\Gamma(1+\mu)\Gamma(K+1)} &\leq 1 - \frac{(K+\mu)^{\mu-1}}{\Gamma(1+\mu)} \\ = 1 - K^{\mu-1}\frac{\left(1+\frac{\mu}{K}\right)^{\mu-1}}{\Gamma(1+\mu)} &\leq 1 - K^{\mu-1}\frac{\left(1+\frac{\mu}{2}\right)^{\mu-1}}{\Gamma(1+\mu)} \\ \overset{*}{\leq} 1 - K^{\mu-1} &= (1-\mu)K^{\alpha}\ln K \leq (1-\mu)\ln K \end{aligned} \tag{11}$$

for some $\alpha \in (\mu-1, 0)$. Here, the estimation $\overset{*}{\leq}$ follows from Lemma 2.5 with $x = 1$. Clearly, inequality (11) is valid also for $k = 1$. □

Remark 1. Solutions x and y from Corollary 3.3 satisfy (9) e.g. in the following cases:

(1) if

$$K \leq 2L_1^{-\frac{1}{\mu}} W\left(\frac{1}{2}L_1^{\frac{1}{\mu}}\sqrt{\frac{\mu}{(1-\mu)^{1-p}}}\right) \tag{12}$$

where W is the Lambert W function [7] defined as the inverse function to $w \mapsto w\,\mathrm{e}^w$. Indeed, since $K \ln K \le K^2$, condition (10) is satisfied if

$$\frac{1}{2} L_1^{\frac{1}{\mu}} K\, \mathrm{e}^{\frac{1}{2} L_1^{\frac{1}{\mu}} K} \le \frac{1}{2} L_1^{\frac{1}{\mu}} \sqrt{\frac{\mu}{(1-\mu)^{1-p}}},$$

which is equivalent to (12).

(2) if

$$K \le \frac{1}{L_1^{\frac{1}{\mu}} + 1} \ln \frac{\mu}{(1-\mu)^{1-p}}.$$

Here we used $K \ln K \le \mathrm{e}^K$.

4. The case $\mu \to 1^+$

In this section, we consider an initial value problem for fractional difference equation (1) with $\mu \in (1,2)$ and given $f \colon \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}^n$, along with a difference equation

$$\begin{aligned} \Delta y(k) &= f(k-1, y(k-1)) + y_1, \qquad k \in \mathbb{N}, \\ y(0) &= y_0, \\ \Delta y(0) &= y_1, \end{aligned} \tag{13}$$

where $x_0, x_1, y_0, y_1 \in \mathbb{R}^n$. Again, we suppose assumption (H). Since the solution y of (13) satisfies the equivalent sum equation

$$y(k) = y_0 + k y_1 + \sum_{j=0}^{k-2} f(j, y(j)), \quad k \in \mathbb{N}_0,$$

we can use Lemma 2.8 and assumption (H) to get the estimation

$$\begin{aligned} \|x(k) - y(k)\| &\le \|x_0 - y_0\| + k\|x_1 - y_1\| \\ &+ \left\| \sum_{j=0}^{k-2} \left(\frac{(k-\sigma(j+2-\mu))^{(\mu-1)}}{\Gamma(\mu)} f(j, x(j)) - f(j, y(j)) \right) \right\| \\ &\le \|x_0 - y_0\| + k\|x_1 - y_1\| \\ &+ \frac{L}{\Gamma(\mu)} \sum_{j=0}^{k-2} (k - \sigma(j+2-\mu))^{(\mu-1)} \|x(j) - y(j)\| \\ &+ M \sum_{j=0}^{k-2} \left| 1 - \frac{(j-\sigma(-\mu))^{(\mu-1)}}{\Gamma(\mu)} \right| \end{aligned}$$

for each $k \in \mathbb{N}_0$. Applying Lemma 2.7 with $m = 2$ yields

$$\begin{aligned}
&\|x(k) - y(k)\| \\
&\leq \left(\|x_0 - y_0\| + k\|x_1 - y_1\| + M \sum_{j=0}^{k-2} \left| 1 - \frac{(j - \sigma(-\mu))^{(\mu-1)}}{\Gamma(\mu)} \right| \right) E_\mu(Lk^\mu)
\end{aligned}$$

for each $k \in \mathbb{N}_0$. So, for $x_0 = y_0$, $x_1 = y_1$, we get

$$\|x(k) - y(k)\| \leq M\theta_\mu(k), \quad \theta_\mu(k) = E_\mu(Lk^\mu) \sum_{j=0}^{k-2} \left| 1 - \frac{(j - \sigma(-\mu))^{(\mu-1)}}{\Gamma(\mu)} \right|$$

for each $k \in \mathbb{N}_0$. Clearly, $\theta_\mu(0) = \theta_\mu(1) = 0$ for any $\mu \in (1, 2)$. Note that the absolute value vanishes for $j = 0$. Furthermore, for $j \in \mathbb{Z}_1^{k-2}$, $k \in \mathbb{N}_3$, we have

$$\frac{(j - \sigma(-\mu))^{(\mu-1)}}{\Gamma(\mu)} = \frac{\Gamma(j + \mu)}{\Gamma(\mu)\Gamma(j + 1)} = \prod_{l=1}^{j} \frac{j + \mu - l}{j + 1 - l} > 1. \tag{14}$$

Hence, analogously to (7) we derive

$$\begin{aligned}
\sum_{j=0}^{k-2} \left| 1 - \frac{(j - \sigma(-\mu))^{(\mu-1)}}{\Gamma(\mu)} \right| &= \frac{1}{\Gamma(\mu)} \sum_{j=1}^{k-2} \frac{\Gamma(j + \mu)}{\Gamma(j + 1)} - \sum_{j=1}^{k-2} 1 \\
&= \frac{1}{\Gamma(1 + \mu)} \left[\frac{\Gamma(k + \mu - 1)}{\Gamma(k - 1)} - \frac{\Gamma(1 + \mu)}{\Gamma(1)} \right] - k + 2 \\
&= \frac{\Gamma(k + \mu - 1)}{\Gamma(1 + \mu)\Gamma(k - 1)} - k + 1, \quad k \in \mathbb{N}_3,
\end{aligned} \tag{15}$$

which remains valid also for $k = 2$.

Theorem 4.1. *Under assumption* (H), *the solution x of* (1) *with $\mu \in (1, 4/3)$ uniformly converges on any set $\mathbb{Z}_0^K$, $K \in \mathbb{N}_2$ to the solution y of* (13) *if $\mu \to 1^+$ and $x_0 = y_0$, $x_1 = y_1$.*

Proof. Note that the limit

$$\lim_{\mu \to 1^+} \left(\frac{\Gamma(k + \mu - 1)}{\Gamma(1 + \mu)\Gamma(k - 1)} - k + 1 \right) = 0$$

is uniform with respect to $k \in \mathbb{Z}_2^K$, since

$$\begin{aligned}
0 &\leq \frac{\Gamma(k + \mu - 1)}{\Gamma(1 + \mu)\Gamma(k - 1)} - k + 1 = (k - 1) \left(\frac{\Gamma(k + \mu - 1)}{\Gamma(1 + \mu)\Gamma(k)} - 1 \right) \\
&\leq (k - 1) \left(\frac{k^{\mu-1}}{\Gamma(1 + \mu)} - 1 \right) \leq (K - 1) \left(\frac{K^{\mu-1}}{\Gamma(1 + \mu)} - 1 \right) \xrightarrow{\mu \to 1^+} 0
\end{aligned}$$

due to Lemma 2.4. Consequently, estimation of $E_\mu(Lk^\mu)$ given by Lemma 2.10 together with identity (15) proves the statement. $\square$

From (14) and (15) one can see that the function

$$\theta_\mu(k) = \begin{cases} 0, & k \in \{0,1\}, \\ \left(\frac{\Gamma(k+\mu-1)}{\Gamma(1+\mu)\Gamma(k-1)} - k + 1\right) E_\mu(Lk^\mu), & k \in \mathbb{N}_2 \end{cases}$$

is increasing on $\mathbb{N}_2$ from 0 to ∞. Therefore, Theorem 4.1 gives the next result.

Theorem 4.2. *Under assumption* (H)*, for any* $\varepsilon > 0$*,* $\mu \in (1, 4/3)$ *there exists* $K \in \mathbb{N}_2$ *such that the solution* x *of* (1) *and* y *of* (13) *with* $x_0 = y_0$*,* $x_1 = y_1$ *satisfy*

$$\|x(k) - y(k)\| \le M\varepsilon, \qquad k \in \mathbb{Z}_0^K.$$

This K *is given as a largest integer such that*

$$\left(\frac{\mathrm{e}^{L^{\frac{1}{\mu}}K}}{\mu} + \frac{4\sqrt{3}\sin\frac{\pi\mu}{2}}{9\mu}\right)\left(\frac{\Gamma(K+\mu-1)}{\Gamma(1+\mu)\Gamma(K-1)} - K + 1\right) \le \varepsilon.$$

Again, we present a simpler corollary.

Corollary 4.3. *Under* (H)*, for any* $\varepsilon > 0$*,* $\mu \in (1, 4/3)$*, solutions* x *of* (1) *and* y *of* (13) *with* $x_0 = y_0$*,* $x_1 = y_1$ *satisfy*

$$\|x(k) - y(k)\| \le M\varepsilon, \qquad k \in \mathbb{Z}_0^K$$

with $K \in \mathbb{N}_2$ *the largest integer satisfying*

$$(\mu - 1)\left(9\,\mathrm{e}^{L^{\frac{1}{\mu}}K} + 4\sqrt{3}\sin\frac{\pi\mu}{2}\right) K^\mu \ln K \le 9\varepsilon\mu.$$

In particular, for $\varepsilon = (\mu-1)^p$ *for any fixed* $p \in (0,1)$*,*

$$\|x(k) - y(k)\| \le M(\mu-1)^p, \qquad k \in \mathbb{Z}_0^K, \tag{16}$$

where $K \in \mathbb{N}_2$ *is the largest integer satisfying*

$$\left(9\,\mathrm{e}^{L^{\frac{1}{\mu}}K} + 4\sqrt{3}\sin\frac{\pi\mu}{2}\right) K^\mu \ln K \le \frac{9\mu}{(\mu-1)^{1-p}}. \tag{17}$$

Proof. From the inequality

$$\begin{gathered} \frac{\Gamma(K+\mu-1)}{\Gamma(1+\mu)\Gamma(K)} - 1 \le \frac{K^{\mu-1}}{\Gamma(1+\mu)} - 1 \\ \le K^{\mu-1} - 1 = (\mu-1)K^\alpha \ln K \le (\mu-1)K^{\mu-1}\ln K \end{gathered}$$

for some $\alpha \in (0, \mu-1)$, we derive

$$\begin{gathered} \frac{\Gamma(K+\mu-1)}{\Gamma(1+\mu)\Gamma(K-1)} - K + 1 \\ \le (\mu-1)(K-1)K^{\mu-1}\ln K \le (\mu-1)K^\mu \ln K. \end{gathered}$$

$\square$

Remark 2. Solutions x and y from Corollary 4.3 satisfy (16) e.g. in the following cases:

(1) if

$$K \leq (\mu+1)L^{-\frac{1}{\mu}} W\left(\frac{1}{\mu+1} L^{\frac{1}{\mu}} \left(\frac{9\mu}{16(\mu-1)^{1-p}}\right)^{\frac{1}{\mu+1}}\right). \tag{18}$$

Indeed, using estimations

$$K^{\mu} \ln K \leq K^{\mu+1}, \qquad 4\sqrt{3}\sin\frac{\pi\mu}{2} \leq 7\,\mathrm{e}^{L^{\frac{1}{\mu}}K}$$

one can show that condition (17) is fulfilled if

$$16\,\mathrm{e}^{L^{\frac{1}{\mu}}K}\, K^{\mu+1} \leq \frac{9\mu}{(\mu-1)^{1-p}},$$

which is equivalent to (18).

(2) if

$$K \leq \frac{1}{L^{\frac{1}{\mu}}+1} \ln \frac{9\mu}{16(\mu-1)^{1-p}}.$$

Here we applied $K^{\mu} \ln K \leq K^2 \ln K \leq \mathrm{e}^K$ for each $K \in \mathbb{N}_2$.

Next, we present a simple illustrative example.

Example 4.4. Let us consider the following initial value problems:

$$\begin{aligned} \Delta_*^{\mu} x(k) &= px(k+\mu-1), \qquad k \in \mathbb{N}_{1-\mu},\\ x(0) &= x_0, \end{aligned} \tag{19}$$

$$\begin{aligned} \Delta y(k) &= py(k), \qquad k \in \mathbb{N}_0,\\ y(0) &= y_0, \end{aligned} \tag{20}$$

$$\begin{aligned} \Delta_*^{\mu} u(k) &= pu(k+\mu-2), \qquad k \in \mathbb{N}_{2-\mu},\\ u(0) &= u_0,\\ \Delta u(0) &= u_1, \end{aligned} \tag{21}$$

$$\begin{aligned} \Delta v(k) &= pv(k-1)+v_1, \qquad k \in \mathbb{N},\\ v(0) &= v_0,\\ \Delta v(0) &= v_1, \end{aligned} \tag{22}$$

where $\mu \in (0,1)$ in (19) and $\mu \in (1,2)$ in (21). It can be shown that the difference equations have the solutions $y(k) = (1+p)^k y_0$ and $v(k) = c_1\lambda_1^k + c_2\lambda_2^k - y_1/p$ with

$$\lambda_1 = \frac{1-\sqrt{1+4p}}{2}, \qquad \lambda_2 = \frac{1+\sqrt{1+4p}}{2},$$
$$c_1 = \frac{1}{2}\left(y_0 + \frac{y_1}{p}\right) - \frac{1}{2\sqrt{1+4p}}\left(2y_1 + y_0 + \frac{y_1}{p}\right),$$
$$c_2 = \frac{1}{2}\left(y_0 + \frac{y_1}{p}\right) + \frac{1}{2\sqrt{1+4p}}\left(2y_1 + y_0 + \frac{y_1}{p}\right).$$

Next,

$$x(k) = x_0 + \frac{p}{\Gamma(\mu)} \sum_{j=0}^{k-1} (k - \sigma(j+1-\mu))^{(\mu-1)} x(j)$$

by [5, Lemma 2.4], and

$$u(k) = u_0 + ku_1 + \frac{p}{\Gamma(\mu)} \sum_{j=0}^{k-2} (k - \sigma(j+2-\mu))^{(\mu-1)} u(j)$$

by Lemma 2.8. To see the convergence, we set all the initial conditions equal to 1, i.e., $x_0 = y_0 = u_0 = v_0 = u_1 = v_1 = 1$, and $p = 0.2$. Figure 1 depicts the convergences $x \to y$ and $u \to v$ as $\mu \to 1^-$ and $\mu \to 1^+$, respectively.

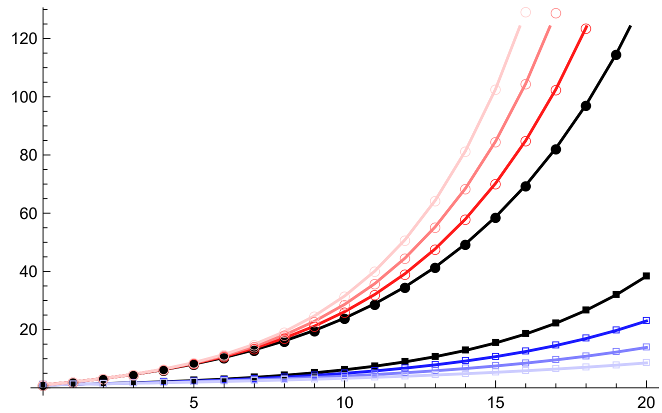

Figure 1. Convergence of solutions of fractional difference equations (19) (blue empty squares), (21) (red empty circles) to solutions of integer-order difference equations (20) (black filled squares), (22) (black filled circles), respectively. The closer $\mu \in \{0.7, 0.8, 0.9, 1.1, 1.2, 1.3\}$ is to 1, the more saturated the colors are.

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