On the uniqueness of solutions to a class of discontinuous dynamical systems

Marius-F. Danca
Avram Iancu University, Department of Applied Sciences, Cluj-Napoca, Romania

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Abstract

The study of uniqueness of solutions of discontinuous dynamical systems has an important implication: multiple solutions to the initial value problem could not be found in real dynamical systems; also the (attracting or repulsive) sliding mode is inherently linked to the uniqueness of solutions. In this paper a strengthened Lipschitz-like condition for differential inclusions and a geometrical approach for the uniqueness of solutions for a class of Filippov dynamical systems are introduced as tools for uniqueness. Several theoretical and practical examples are discussed.

keywords: Filippov solutions, attractive sliding mode, repulsive sliding mode, strengthened one side Lipschitz condition.

1 Introduction

Over the years, there has been growing interest and need for the modeling, analysis, and control of non-smooth dynamical systems characterized by discontinuous changes in system properties. A particular case is represented by dynamical systems discontinuous with respect to the state variable. Examples of such phenomenon are exhibited by various mechanical, electrical, biological, and natural systems.

Circa 1781 Coulomb introduced a model of the friction contact between solid bodies [1]. The Coulomb model while widely used and accurate enough in many engineering applications it gives rise to difficult computational and analytical problems. Actually the discontinuous nature of Coulomb friction makes the general problem difficult. Many processes in industry and elsewhere exhibit regime switches, which may be described as fast phenomena that may lead to large changes in the system dynamics and/or the system state. Such switches may be due to external causes or may occur as a result of the process dynamics itself. Examples include electrical networks in which for instance thyristors switch from conducting to non-conducting mode, robotic mechanisms which switch from compliant to non-compliant modes, processes under the influence of a discrete (for instance on/off) controller, thermostats implement closed-loop bang-bang controllers to regulate room temperature, aerial and underwater terrains are yet two more examples where discontinuities naturally occur from the interaction with the environment.

Regime-switching systems occur in several disparate areas and are also known as discontinuous dynamical systems, multimodal systems, systems with variable topology, or hybrid systems. Although one may attempt to describe the involved fast phenomena in a continuous way, this easily leads to stiff numerical problems and an alternative is to model the switching as discrete events. Examples and background theory on this field can be found in the early works [2][3], [4], [5][6], [7], or in recent works as [8], [9] or[10].

Since the vector field is discontinuous in such a system, continuously differentiable curves that satisfy the system do not exist in general, and we must face the issue of identifying a suitable notion of solution. A look into the literature reveals that there is not a unique answer to this question. Depending on the specific problem at hand, different authors have different definitions.

Although general notions and basic elements of the theory of such systems may be found in the early references, the book of Filippov [11] is unanimously accepted today as one of the major contributions to the general theory of discontinuous dynamical systems.

For simplicity, we consider vector fields defined over the whole Euclidean space. Thus, we focus on dynamical systems which can be modelled by the following autonomous Initial Value Problem (IVP) discontinuous with respect to the state variable

$$\dot{x} = f(x), \quad x(0) = x_0, \quad x_0 \in \mathbb{R}^n, \quad t \in I = [0, \infty), \tag{1}$$

under the standing assumption that $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded in the following form

$$\dot{x} = f\ (x) := A\,x + B\ s(x), \quad x(0) = x_0, \quad t \in I = [0,\, \infty)\,, \tag{2}$$

with $A = (a_{i,j})_{n \asymp n}\,, B = (b_{i,j})_{n \asymp n}$ real constant matrices and the vector function $s : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$s(x) = \begin{pmatrix} sgn(x_1) \\ \vdots \\ sgn(x_n) \end{pmatrix}.$$

This particular form of IVP describes a lot of switching physical phenomena[1].

In general, the great generality of known discontinuous mechanical systems analyzed in the literature are planar while discontinuous electrical systems are three-dimensional autonomous [2]

Throughout this paper, for the sake of simplicity, only ODEs without the initial conditions will be considered.

Example 1 *The simplest example of Coulomb friction has the following model [14]*

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 - sgn(x_2), \end{aligned} \tag{3}$$

with $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ $s(x) = \begin{pmatrix} sgn(x_1) \\ sgn(x_2) \end{pmatrix}$.

Example 2 *The autonomous harmonic oscillator (a mechanical system with two degrees of freedom) is one of the most popular examples in textbooks of periodic behavior in physical systems. A nonsmooth version [15] is modeled by*

[1] A more general form of IVP (2) modeling non-smooth dynamical systems is analyzed in [12].

[2] However there are hygher dimensional discontinuous mechanical dynamical systems presented in literature (see e.g. [13]).

$$\begin{aligned}\dot{x}_1 &= sgn(x_2),\\ \dot{x}_2 &= -sgn(x_1),\end{aligned} \tag{4}$$

In the following examples, a represents the bifurcation parameter whose variation influences the system behavior.

Example 3 *In [16] several variants of the mathematical model of Chua's circuit are presented . One of them is the following (see [17] for some properties of this system)*

$$\begin{aligned}\dot{x}_1 &= -2.57x_1 + 9\,x_2 + 3.87\ sgn\,(x_1),\\ \dot{x}_2 &= x_1 - x_2 + x_3,\\ \dot{x}_3 &= -\alpha\, x_2\,,\ \ \alpha > 0.\end{aligned} \tag{5}$$

Example 4 *One of the simplest possible structure for a chaotic oscillator has the following mathematical model [18]*

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= x_3,\\ \dot{x}_3 &= -\alpha(x_1 +\ x_2 + x_3 - sgn(x_1))\,,\ \ \alpha > 0.\end{aligned} \tag{6}$$

Example 5 *The following system represents a simplified model of the regulation systems of a steam turbine [19]*

$$\begin{aligned}\dot{x}_1 &= \alpha\ (x_3 - x_1 -\ sgn\,(x_2))\,,\\ \dot{x}_2 &= x_1 - x_2, \qquad\qquad\qquad .\\ \dot{x}_3 &= -\ x_2,\ \alpha > 0.\end{aligned} \tag{7}$$

Example 6 *The next theoretical example is taken from [11]*

$$\begin{aligned}\dot{x}_1 &= 2sgn(x_1) - 6sgn(x_2) - 2sgn(x_3),\\ \dot{x}_2 &= 6sgn(x_1) - 4sgn(x_3),\\ \dot{x}_3 &= 12sgn(x_1) + sgn(x_2) - 9sgn(x_3).\end{aligned} \tag{8}$$

The existence and uniqueness of solutions to IVP (2) are essential to define the notion of dynamical systems especially for discontinuous dynamical systems, because due to the right-hand discontinuity, classical solutions of IVP might not even exist and even Caratheodory's existence theorem (see e.g. [20]) fails to apply.

As an example let us consider the discontinuous IVP: $\dot{x} = sgn\,(x)$, $x(0) = 0$. There is no classical solution starting from 0 despite the fact that $x = 0$ verifies the equation, because in some small neighborhood of $x = 0$ this solution presents the tendency to "jump" to one of the possible solutions $x(t) = \pm t$.

However ODE: $\dot{x} = -\ sgn(x)$ has a solution starting from any initial condition $x(0)$

$$x(t) = \begin{cases} x(0) + t & \text{for } x < 0 \text{ defined on } [0, -x(0)), \\ 0, & \text{for } x = 0 \text{ defined on } [0, \infty), \\ x(0) - t & \text{for } x > 0 \text{ defined on } [0, x(0)), \end{cases}$$

but these solutions (in the classical sense, i.e. continuously differentiable) cannot be continued along the axis $x = 0$ because the derivative has a discontinuity in a small neighborhood of $x = 0$.

The following ODE

$$\dot{x} = 2 - 3\, sgn(x), \tag{9}$$

has, for $x \neq 0$, the solutions

$$x(t) = \begin{cases} 5\,t + C_1, & x < 0, \\ -t + C_2, & x > 0, \end{cases}$$

but, as t increases, these classical solutions tend to the line $x = 0$, where they cannot continue to evolve since the function $x(t) = 0$ does not verifies the equation. Thus, there are no classical (continuously differentiable) solutions starting from 0 (see Fig.1a).

Thus, one can see that the discontinuity does not necessarily implies the non-existence of solutions. To provide the possibility for the solutions to IVP (2) to be continued it is necessary to modify its right-hand side.

The structure of the paper is as follows: Section 2 presents the set-valued IVP associated with IVP 2; Section 3 deals with the existence of solutions to the set-valued IVP; Section 4 presents the main results of the paper regarding the uniqueness of the generalized solutions and Section 5 shows several examples. The last section gives some concluding remarks.

2 The set-valued IVP

For discontinuous vector fields, existence and uniqueness of solutions is not guaranteed in general no matter what notion of solution is chosen. Also, the classical notion of solution for ordinary differential equations is too restrictive when considering discontinuous vector fields. A resolution of the difficulty about existence is to extend the notion of differential equation to differential inclusion. The problem was solved by Filippov [11] using a generalized concept of solution. The IVP is shifted to the following set-valued one

$$\dot{x} \in F(x),\ x(0) = x_0, \text{ for almost all } t \in I\,, \tag{10}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued function mapping $\mathbb{R}^n$ into the set of all nonempty, compact and convex subsets of $\mathbb{R}^n$ and is obtained by the so-called *Filippov regularization*

$$F(x) = \bigcap_{\varepsilon > 0} \bigcap_{\mu\,(M) = 0} \overline{conv}\, f\left((x + \varepsilon\, B) \setminus M\right), \tag{11}$$

where M is the set of discontinuity points of f, B the unit ball in $\mathbb{R}^n$, μ the Lebesgue measure and $\overline{conv}$ the closed convex hull. At the points where the function f is continuous, $F(x)$ consists of one point, which coincides to the value of f at this point, i.e. $F(x) = \{f(x)\}$. At the discontinuity points, the set $F(x)$ is given by (11). In other words, $F(x)$ is the convex hull of values of $f\,(x^*)$, $x^* \in M$, ignoring the behavior on null sets. In Filippov's definition, the crucial point is the "minus a null set M" (see (11)):

the concept is set up to ignore possible misbehavior of f on sets of small measure. This approach is implicitly used in most introductory references.

Thus, the state derivative belongs to a set of directions, rather than being a specific direction determined by the vector field (Fig.2). This flexibility is key to providing general conditions on the vector field under which Filippov solutions exist.

Because of the way the Filippov set-valued map is defined, its value at a point is actually independent of the value of the vector field at that specific point.

As an example, the Filippov regularization applied to the unidimensional *sign* function gives the set-valued function

$$Sgn\,(x) = \begin{cases} \{-1\} & x < 0, \\ [-1,\,1] & x = 0, \\ \{+1\} & x > 0. \end{cases}$$

Thus, $sgn(0)$ is taken as the whole interval $[-1, 1]$ "connecting" the points -1 and $+1$.

Related versions of the Filippov regularization were developed by Krasovskij, Hermes, and others (see review papers [21] and [23]).

Applying the Filippov regularization to the IVP (2) transforms it into a multivalued Cauchy problem

$$\dot{x} \in F(x) := Ax + B\ S(x), \quad x(0) = x_0, \text{ for almost all } t \in I, \tag{12}$$

where $S(x) = (Sgn(x_1), \ldots, Sgn(x_n))^T$. The discontinuity hypersurface (manifold) is a polyhedron Σ whose flat faces of equations $\phi(x_i) := b_{ij}x_i = 0$ split $\mathbb{R}^n$ into several domains $\Omega_i^{\pm}$ generated by the sign of ϕ, i.e. $\mathbb{R}^n = \Omega^- \bigcup \Omega^+ \bigcup \Sigma$ where $\Omega^- = \bigcup \Omega_i^-$ and $\Omega^+ = \bigcup \Omega_i^+$.

For example, (5) becomes

$$\begin{aligned} &\dot{x}_1 \in -2.57x_1 + 9\,x_2 + 3.86\ Sgn\,(x_1), \\ &\dot{x}_2 = x_1 - x_2 + x_3, \\ &\dot{x}_3 = -\alpha\,x_2\,,\ a > 0, \end{aligned} \tag{13}$$

for which the discontinuity surface Σ has the equation $\phi(x_1) := x_1 = 0$. The domains generated by this surface Σ are $\Omega^+ = \{x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 > 0\ \}$ and $\Omega^- = \{x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 < 0\ \}$.

It is interesting to see that on Ω_i^+(Ω_i^-) the motion is governed by the equation (2) while on Σ by a particular equation deduced in [11], which wil not be discussed in this paper.

2.1 Existence of generalized (Filippov) solutions

Definition 7 *A* generalized (Filippov) solution *to (1) is an absolutely continuous function* $x : [0, \infty) \longrightarrow \mathbb{R}$ *satisfying (10) for almost all* $t \in [0, \infty)$.

Note that any vector field that differs from the vector field of f by a set of measure zero has the same set-valued map, and hence the same set of Filippov solutions.

For our class of IVP (2), a generalized solution is an absolutely continuous function satisfying (12) for almost all $t \in I$.

Non-existence of solutions implies that the model is inherently wrong, so in science the physical system does not cease to exist when a situation where non-existence arises in the model.

On the other hand, the multiplicity of solutions may indicate that there is insufficient information in the model to uniquely predict the outcome of a situation (see [24]).

Therefore, the existence of solutions represents an important task.

The existence of a solution to the IVP (10) is enssured by the well-known existence theorem, presented in many works (e.g. [25], [26], [27]).

Definition 8 *A set-valued function F verifies a* growth condition *if there exist nonnegative constants k and a such that*

$$\| \xi \| \leq k \| x \| + a,$$

for all $\xi \in F(x),\ x \in \mathbb{R}^n$.

Notation 9 *Denote by $\mathcal{F}$ the set of all set-valued functions satisfying the following so-called basic conditions: nonemptyness, closedness and convex-valued that together verifying supplementary the growth condition.*

Theorem 10 *The IVP (2) admits at least one generalized solution.*

Proof. In [17] it is proved that a more general than (12) IVP, with a nonlinear Lipschitz vector function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ instead of Ax in (12), and has in the set-valued right-hand side a function belonging to $\mathcal{F}$. Ax as a linear function is Lipschitz. Thus, F in (12) belongs to $\mathcal{F}$, so (12) admits at least one generalized solution and therefore IVP (2) admits a generalized solution. ■

The generalized solution is continuable to ∞ (i.e. the maximal existence interval is $[0, \infty)$) if a compactness condition is satisfied (see [17], where it is proved that this class of dynamical systems satisfy the condition).

The growth condition implies that all solutions remain in some bounded subset and are being used instead of the global boundness of the right-hand side.

The notions and basic results on the existence of solutions to differential inclusions can be found in [25] and [26].

Remark 11 *i) The closedness condition in Existence Theorem 10 refers to the graph of F (see e.g. [26]).*
ii) The nonemptyness condition guarantees the existence of a global solution on the whole interval I.

If we consider the corresponding set-valued IVP: $\dot{x} \in Sgn\,(x),\quad x(0) = x_0$, then there are multiple Filippov solutions: $x(t) = 0$ for $x_0 = 0$ and $x(t) = \pm t + x_0$ for $x_0 \neq 0$ defined on the maximal interval $[0, \infty)$, while for the set-valued IVP of the discontinuous ODE $\dot{x} = -sgn\,(x)$, the line $x = 0$ is now the unique Filippov solution, defined on the maximal interval $[0, \infty)$. After Filippov regularization, the Filippov solution to ODE (9) can be continued uniquely with $x = 0$.

Remark 12 *Dealing with dynamical systems that are modeled by differential inclusions and not differential equations is only a technical problem, since these systems can be approximated by smooth ones (see [28]).*

3 Uniqueness of Filippov solutions

In general, discontinuous dynamical systems do not have unique Filippov solutions. Multiplicity of solutions indicates that there are insufficient informations on the physical model to uniquely predict its behavior

For the class of IVP (2) there are sufficient conditions for uniqueness of solutions. In this paper, we focus on one-sided Lipschitz-like and geometrical conditions from the vector fields approach.

The existence and uniqueness in Ω_i is enssured in the usual way since in Ω_i, f is continuous. Therefore a special attention should be paid on Σ, where f is discontinuous.

The uniqueness of solution is useful not only for the convergence study of numerical methods for differential inclusions but also to verify if a discontinuous IVP defines a dynamical system or a generalized one (see [29] for the continuous case and [30] for a class of discontinuous dynamical systems).

Generally, Lipschitz-like conditions are utilized to ensure the convergence of difference methods for differential inclusions. Implicitly, in these cases the uniqueness must be assured.

In the cases of set-valued IVP like (12) which modells discontinuous dynamical systems, the usual Lipschitz condition (for set-valued right-hand side) is not adequate. Instead, the uniform one-sided Lipschitz (UOSL) condition is utilized.

Definition 13 *[31] Let $\langle\cdot|\cdot\rangle$ be the scalar product in $\mathbb{R}^n$ with the induced norm $\|\cdot\|$. A set-value function F satisfies an UOSL condition with the one-sided Lipschitz constant λ (not necessarily positive) if*

$$\langle \xi' - \xi'' | x' - x'' \rangle \leq \lambda \parallel x' - x'' \parallel^2,$$

holds for all $x', x'' \in \mathbb{R}^n$, $\xi' \in F(x')$, $\xi'' \in F(x'')$.

However, for our class of discontinuous IVP, we shall use a strengthened version, the Strengthened One-Sided Lipschitz (SOSL) condition, introduced in [31] and [33].

Definition 14 *The set-valued function F satisfies a SOSL condition with one-sided Lipschitz constants $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, if the implication*

$$x_i'' > x_i' \implies \xi_i' \leq \xi_i'' + \lambda_i \parallel x' - x'' \parallel,$$

is true for all $x', x'' \in \mathbb{R}^n$, $\xi' \in F(x')$, $\xi'' \in F(x'')$, and all components $i = 1, \ldots, n$.

Remark 15 *i) For $n = 1$, SOSL condition is satisfied if and only if the usual UOSL condition holds, while for $n > 1$, SOSL condition is stronger then UOSL.*
ii) The unidimensional set-valued function $-Sgn(x)$ satisfies SOSL, while $+Sgn(x)$ does not.

Following some convergence results on numerical methods for differential inclusions presented in several works (e.g. [35] and [22]), the following corollary for the general case of IVP (10) can be established.

Corollary 16 *The IVP (10) with F satisfying SOSL condition admits at most one generalized solution.*

Proof. (sketch) As stated in Remark 15, the SOSL condition implies UOSL condition (see e.g. [35][27] or [11] for a proof) and the IVP (10) satisfying UOSL admits a unique solution. ■

The following theorem, which is the first main result of this paper, provides a sufficient condition for uniqueness.

Theorem 17 *Let the IVP (2). If B is not positive-definite, then the IVP (2) admits a unique (right) generalized solution.*

Proof. The existence comes from Theorem 10. Thus it is easy to verify that F defined by the right-hand side of (12) is of $\mathcal{F}$ class. If B is not positive definite matrix, then $b_{ij}Sgn\left(x_i\right)e^i$, where e^i is the i-th canonical unit vector in $\mathbb{R}^n$, verifies the SOSL condition (see [35]). Ax is Lipschitz continuous and in [31] it is proved that the sum of a Lipschitz function and a function having the form $b_{ij}Sgn\left(x_i\right)e^i$ verifies UOSL condition. Therefore, the uniqueness is enssured (see [17] for the complete proof for a more general case). ■

The SOSL condition is only sufficient, so nothing can be said if at least one of the coefficients of the matrix B is positive. For example, the following ODEs (modified version of an example presented in [14])

$$\begin{cases} \dot{x}_1 = 4 + 2sgn(x_2), \\ \dot{x}_2 = 2 - 4sgn(x_2), \end{cases} \tag{14}$$

has a unique solution even the matrix B contains positive coefficients. In these kind of situations, the geometrical approach could be applied.

To study the motions of dynamical systems modeled by (2) on Σ, denote by N_i the normal vector at a point x belonging to one of the face, Σ_i, of the hypersurface Σ, $x \in \Sigma_i$, just at the boundary of two adjacent domains Ω_i^+ and Ω_i^- for some $i \in \{1, \ldots, n\}$, and let ${f_i}^+$ and ${f_i}^-$ be the vector fields approaching x from Ω_i^+ and Ω_i^- i.e. $\phi(x_i) > 0$ and $\phi(x_i) < 0$, respectively (it is supposed that the positive sense of N_i is directed towards Ω_i^+) and ${f_i}_{N_i}^{\pm}$ the projections of $f^{\pm}$ on N_i.

For the sake of simplicity in the following we will drop the index i.

Then, three possible situations are identified in the following result, which is the second main result on uniqueness presented in this paper.

Theorem 18 *Let IVP (12) with F defined by (11). If, at some point $x \in \Sigma$, one of the following conditions holds*

$$1)\ f_N^- \ f_N^+ > 0,$$

$$2)\ f_N^- > 0 \ \textit{and}\ f_N^+ < 0,$$

then the generalized solution is unique.
If

$$3)\ f_N^+ > 0 \ \textit{and}\ f_N^- < 0,$$

then a generalized solution starting from Σ is not unique.

Proof. (sketch) Under the stated assumptions, when reaching Σ, Filippov solutions might cross it or slide along it. Thus:
1) If f_N^- and f_N^+ have the same sense as N (Fig.3a) a solution starting from the initial condition $x_0 \in \Omega^+$(or Ω^-) will cross the discontinuity surface Σ.The solution stays on the hypersurface at only one instant

of time but not on an interval of time. Thus, the uniqueness assured in $\Omega^{\pm}$ is preserved when Σ is crossed.
2) If f_N^+ has opposite sense to N while f_N^- has the same sense to it (Fig.3b), a generalized solution reaching some $x \in \Sigma$ cannot escape and will be forced to remain on Σ which is attracting.

3) If the generalized solution starts near Σ it will move away from it by the vector fields $f^{\pm}$, but a solution starting from some $x \in \Sigma$ may at one moment of time either leave Σ to Ω^+ or to Ω^- (Fig.3c) or stay for some time interval on Σ, which is repelling (Fig.3d). ■

These motions corresponding to 2) and 3) are called *attractive sliding mode* and *repulsive sliding mode,* respectively. The repulsive sliding mode cannot occur in real systems (as stated in [11]), however.

More information about the vector field approach of uniqueness of solutions is referred to [11].

Remark 19 *i) Table 1 presents a comparation (advantages and disavantages) between Theorem 17 and Theorem 18.*

	Theorem 17	*Theorem 18*
accessibility	+	-
phase portrait informations	-	+
sliding mode informations	-	+
solution multiplicity	-	+
global information	+	-

$Table 1$

ii) Because the nonpositivenes of B in Theorem 17 is only a sufficient-like condition, if some coefficients $b_{ij} \in B$ are positive, then nothing can be said about the uniqueness and therefore Theorem 18 can be explored as well.
iii) The geometric approach in Theorem 18 has a disavantage since it refers to a single point. Anyway in our case, if one of the conditions 1), 2) or 3) is satisfied at some point of a discontinuity surface, then the condition is fulfilled over the entire surface.
iv) For the attracting sliding mode, if we consider the backward time, then the motion becomes a repulsing sliding mode, i.e. the (left) generalized solution may not be unique.
v) Sliding mode in case 1) of Theorem 18 is impossible.
vi) In case 3), the classical solutions of a discontinuous IVP cannot cross the discontinuity surface Σ, but once the IVP is switched into a set-valued IVP by e.g. Filippov regularization, the "missing" values between the segment ends f^+ and f^- are filled and a generalized solution from e.g. Ω_i^+ near Σ can change the direction to Ω_i^- and cross Σ at some x, because this derivative enjoys enough values given by $F(x)$ which link the end points of f^+ and f^- (Fig.2).

In summary, the possibilities depicted in Scheme 4 may actually happen.

4 Applications

All the presented examples verify the existence conditions in Theorem 10. Therefore in this section we shall study only the uniqueness of solutions.

The numerical solutions presented in this section were obtained using a special numerical method for differential inclusions, namely forward Euler method (the background for numerical method for differential inclusions can be found in [32][34][36] or [35]).

- First consider the example modeled by (3) which has a unique solution since $b_{22} < 0$ and Theorem 17 applies. Also, $f^{-}(x) = \lim f(x)$ for $(x_1, x_2) \longrightarrow (x_1, 0)$ with $x_2 < 0$ has the expressions $f^{-}(x) = (x_2, -x_1 + 1)$ and $f^{+}(x) = (x_2, -x_1 - 1)$. The projections of $f^{\pm}$ on $N = (0, 1)$ are $f_N^{-}(x) = -x_1 + 1$ and $f_N^{+} = -x_1 - 1$. Therefore: a) for $x_1 \notin (-1,\ 1)$, f_N^{-}, f_N^{+} have the same sign and condition 1) in Theorem 18 is certified; b) $x_1 \in (-1,\ 1)$, $f_N^{-} > 0$ and $f_N^{+} < 0$ and the attracting sliding mode appears; the unique solution tends to the origin (Fig.5).

- Second, consider the Example modeled by (4). Because $b_{12} > 0$, Theorem 17 cannot be used. Therefore, we will check Theorem 18 (Fig.6). The lines $x_1 = 0$ and $x_2 = 0$ split the plane in four domains I, II, III, IV, separated by the lines $\Sigma_1, \Sigma_2, \Sigma_3$ and Σ_4 with the normals $N_1 = (0, 1)$ and $N_2 = (1, 0)$. For example in the domain I, $x_1, x_2 > 0$ the vector field is $f_I = (1, -1)$ and across Σ_2 we have $f_I N_2 = 1 > 0$. Therefore, we have the case 1) of Theorem 18 and the solutions is unique. The same happens across all the discontinuity lines.

- In the case of Example (5), again B is not nonpositive and therefore the uniqueness has to be analyzed via vector field directions. A chaotic trajectory for $\alpha = 15.7$ is depicted in Fig.7a. $f^{-}(x) = \lim f(x)$ for $x \to (0, x_2, x_3)$, $x_1 < 0$. Thus $f^{-}(x) = (9x_2 - 3.87, -x_2 + x_3, -ax_2)^T$. Also $f^{+}(x) = (9x_2 + 3.87, -x_2 + x_3, -ax_2)^T$. The normal to the discontinuity surface Σ of equation $x_1 = 0$ is $N = (1, 0, 0)$. Thus $f_N^{-} = f^{-}N = -3.87 < 0$ while $f_N^{+} = 3.87 > 0$. Therefore Theorem (18, 3), shows that the solution is not unique. In Fig.7b, it can be seen that in the neighborhood of the discontinuity surface the vector fields have opposite sense.

- The example (6) has $b_{33} > 0$, therefore the uniqueness has to be verified with Theorem18. It is easy to see that $f_N^{\pm}$ have opposite signs in the neighborhood of Σ, and thus the generalized solution is not unique (Fig.8).

- For the example (7), B is negative and thus by (17) the solution is unique. This can be deduced via Theorem 18 too. Thus $f^{-}(x) = (ax_3 - ax_1 + a, x_1, 0)^T$, $f^{+}(x) = (ax_3 - ax_1 - a, x_1, 0)^T$, $N = (0, 1, 0)$, $f_N^{+} = f_N^{-} = x_1$ and the condition 1) in Theorem 18 is verified (Fig.9).

- Finally consider the example (8). Theorem 17 cannot be used. Therefore, the geometrical approach via Theorem 18, 1) will be used. As can be seen in Fig.10, on each discontinuity surfaces the vector fields have the same sense, so the generalized solution is unique.

5 Concluding remarks

In this paper we have presented in a unified way some sufficient conditions on the uniqueness of solutions to the discontinuous IVP 2. The approach consists in the use of a strengthened one-sided Lipschitz condition utilized for the convergence of some special numerical methods for differential inclusions, and the vector field approach which can be used as an alternative for the case when the above Lipschitz condition fails.

Despite the fact that Theorem 17 is easily accessible in applications -it requires solely the negativeness of the b coefficients in the IVP (2)- it is restrictive enough. Thus, there are cases when some b coefficients are positive and yet the Filippov solutions are unique. Theorem 18 cope with these situations even it works in a more laborious way.

Theorem 18 allows the study of attractive and repulsive sliding modes, which are essential for control techniques (e.g. *sliding mode control* in higher-order systems).

The study of several examples coming from theoretical or practical applications shows that this algorithm for the uniqueness (Figure 4) is very useful.

A similar way for the uniqueness of a more general class of systems remains as a future task.

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Figure captions

Figure 1: a) The IVP (9): a) classical solutions; b) Filippov solutions.

Figure 2: Vector fields (sketch): a) before regularization; b) after regularization.

Figure 3: Sketch of discontinuous vector fields: a) the unique solution crosses the discontinuity surface; b) attracting sliding mode: the unique solution once arived on Σ slides along it; c) and d) repulsing sliding mode: c) the solution (not unique) remains on Σ only for an instant of time and may go to any of the continuity domains; d) the solution remains on Σ for an interval of time.

Figure 4: The algorithm of the uniqueness study via Theorems 17 and 18.

Figure 5. Phase portrait and vector fields for example 1. The Filippov solution is unique. Between -1 and +1 there is an attracting sliding mode.

Figure 6. Phase portrait and vector fields for example 2. The Filippov solution is unique.

Figure 7. Phase portrait and vector fields for the generalized Chua discontinuous dynamical system modele by (5): a) three-dimensional view; b) details wherefrom the non-uniqueness can be deduced: the vector fields near Σ have oposite signs.

Figure 8. Phase portrait and vector fields for the electronic oscillator described in Example 4. The vector fields nier Σ have oposite signs ($f^{\pm}(x) = (x_2, x_3, -ax_2 - ax_3 \pm a)$) and a repulsive sliding mode appears.

Figure 9. Phase portrait and vector fields for the turbine modeled by (7). The vector fields have the same orientation near the discontinuity surface therefore, the solution is unique.

Figure 10. The theoretical example 6. a) phase portrait. b) vector fields and discontinuity planes.

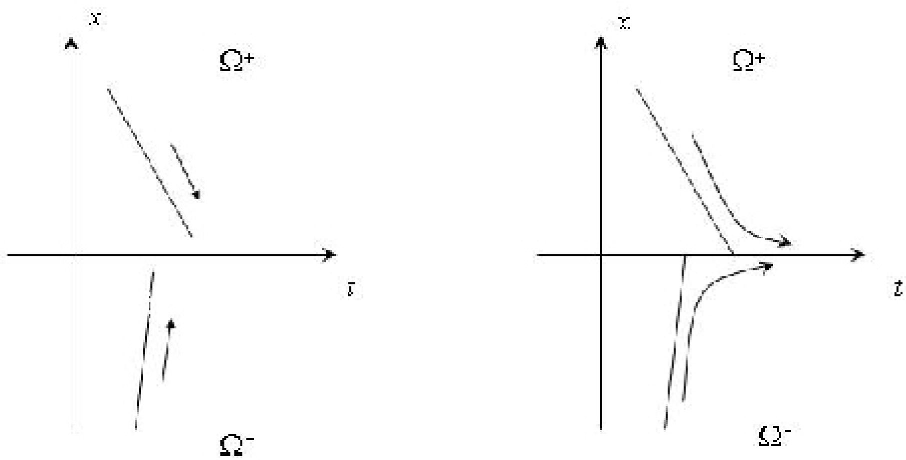


Figure 1

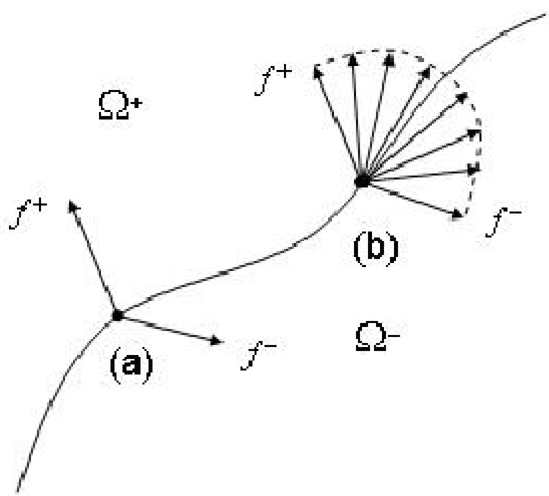


Figure 2

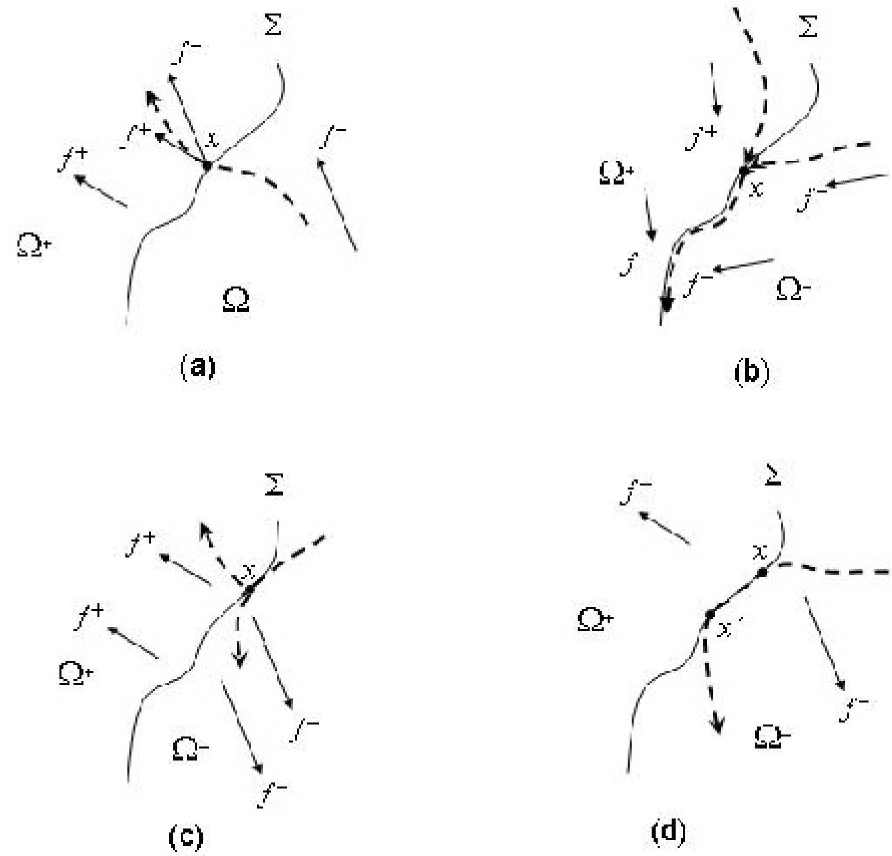


Figure 3

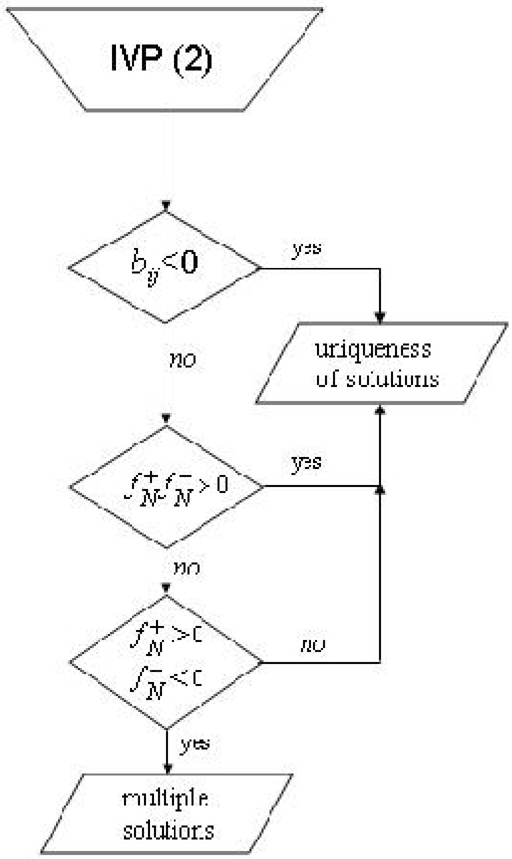

IVP (2)
$b_{ij}<0$
yes
uniqueness of solutions
no
$f_N^+ f_N^- > 0$
yes
no
$f_N^+ > 0$
$f_N^- < 0$
no
yes
multiple solutions

Figure 4

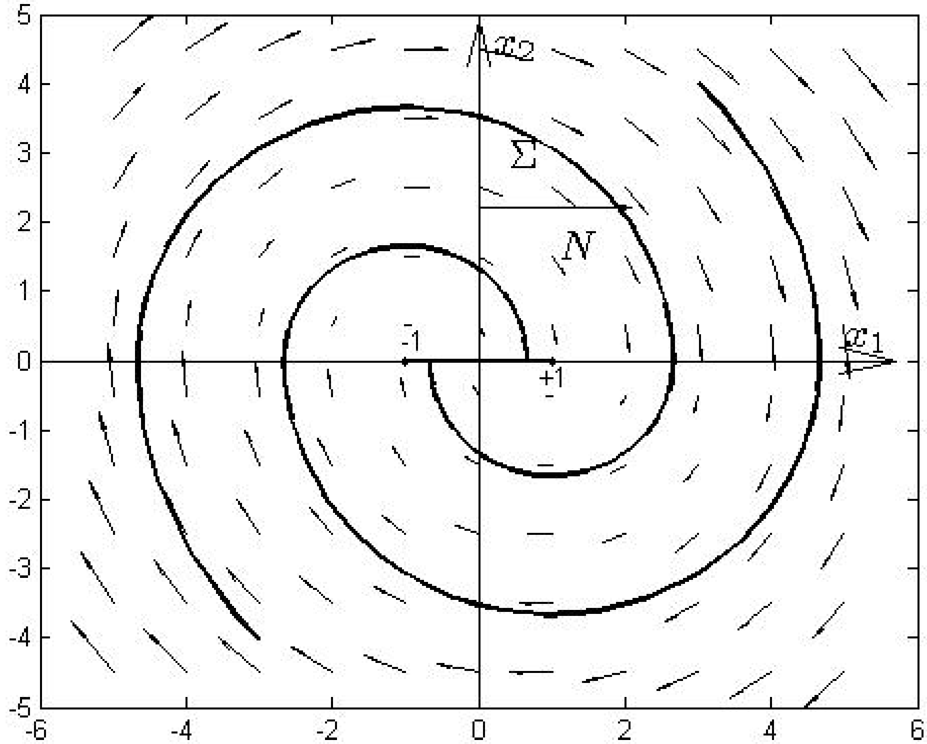
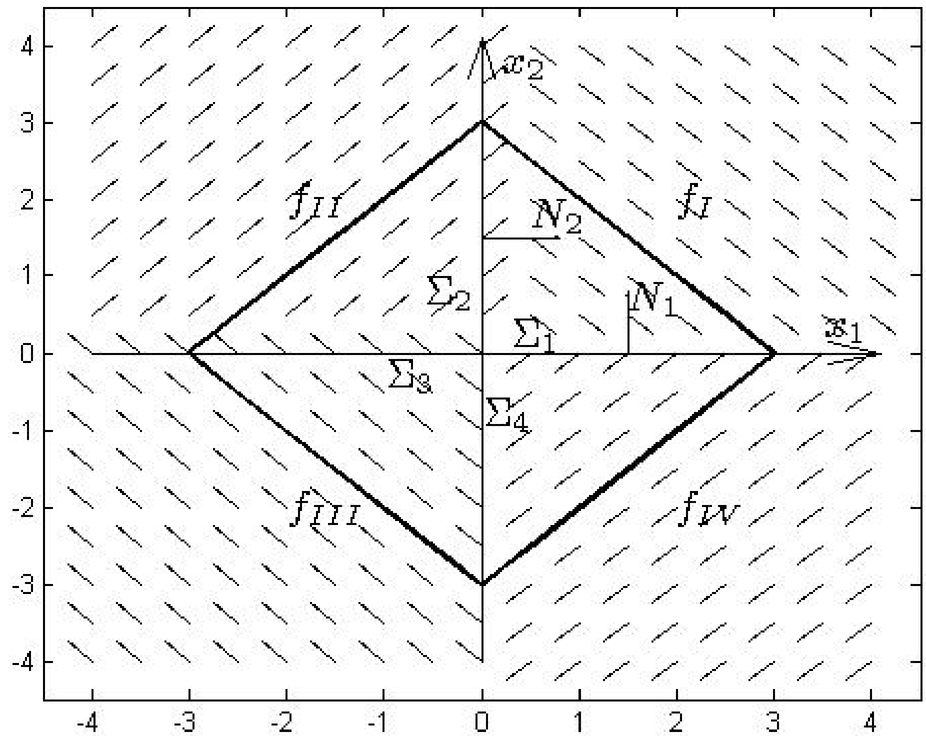


Figure 5



4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4
x_2
x_1
f_I
f_{II}
f_{III}
f_{IV}
N_1
N_2
Σ_1
Σ_2
Σ_3
Σ_4

Figure 6

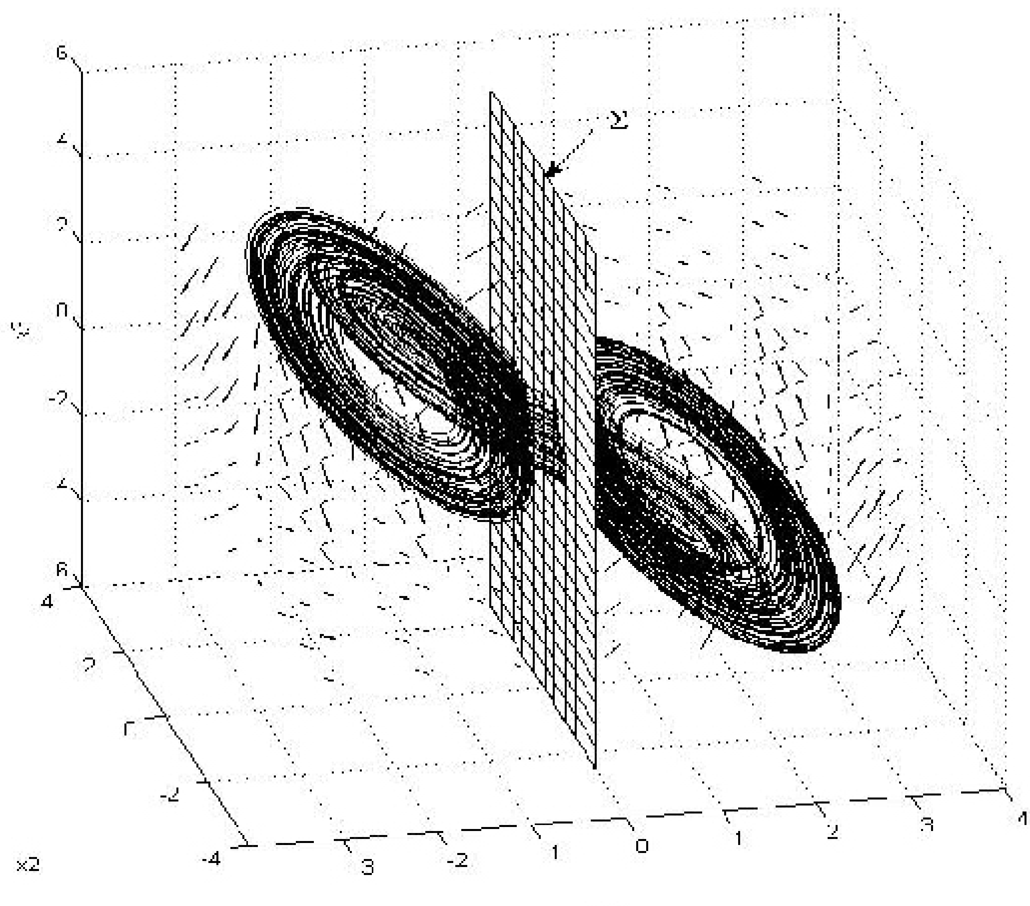


Figure 7a

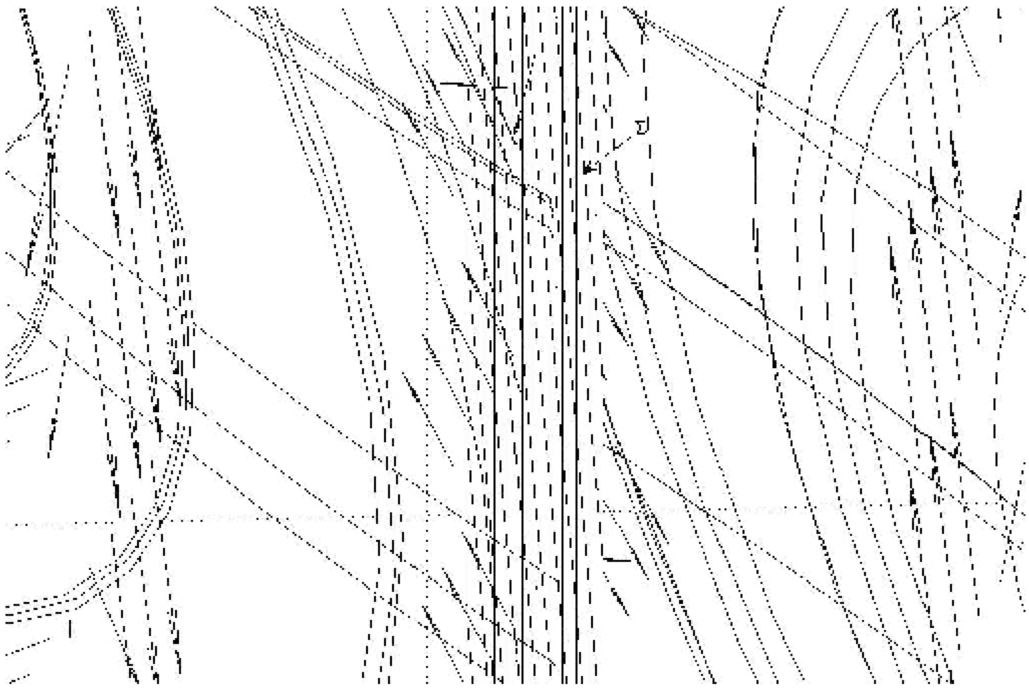


Figure 7b

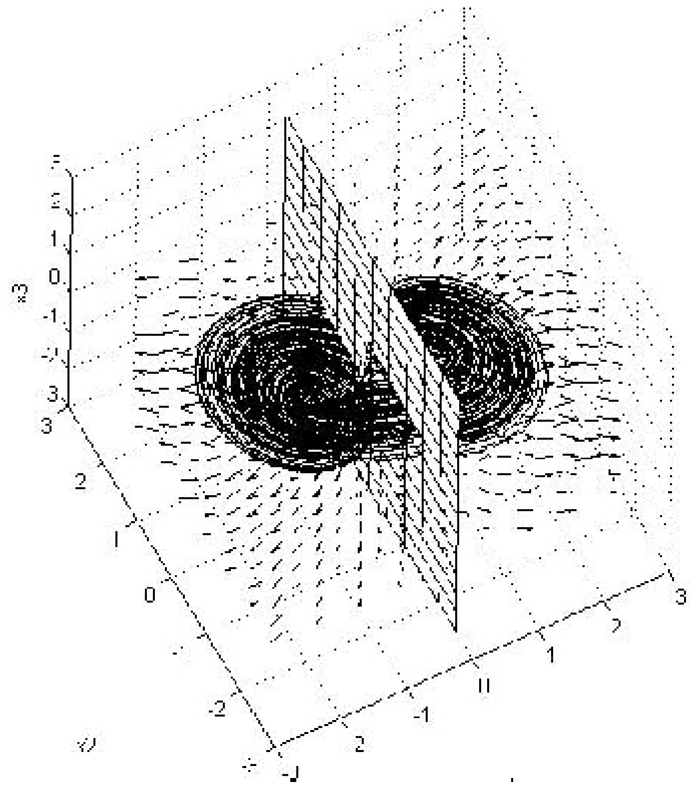


Figure 8

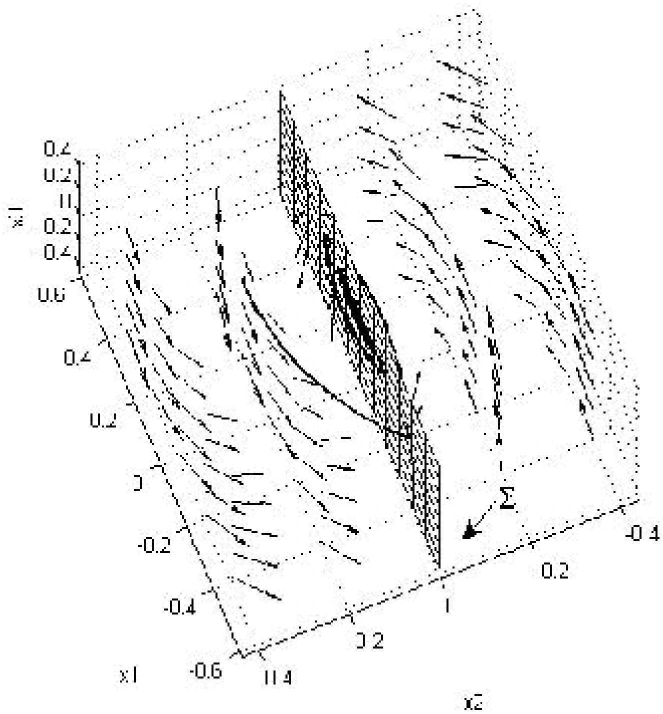


Figure 9

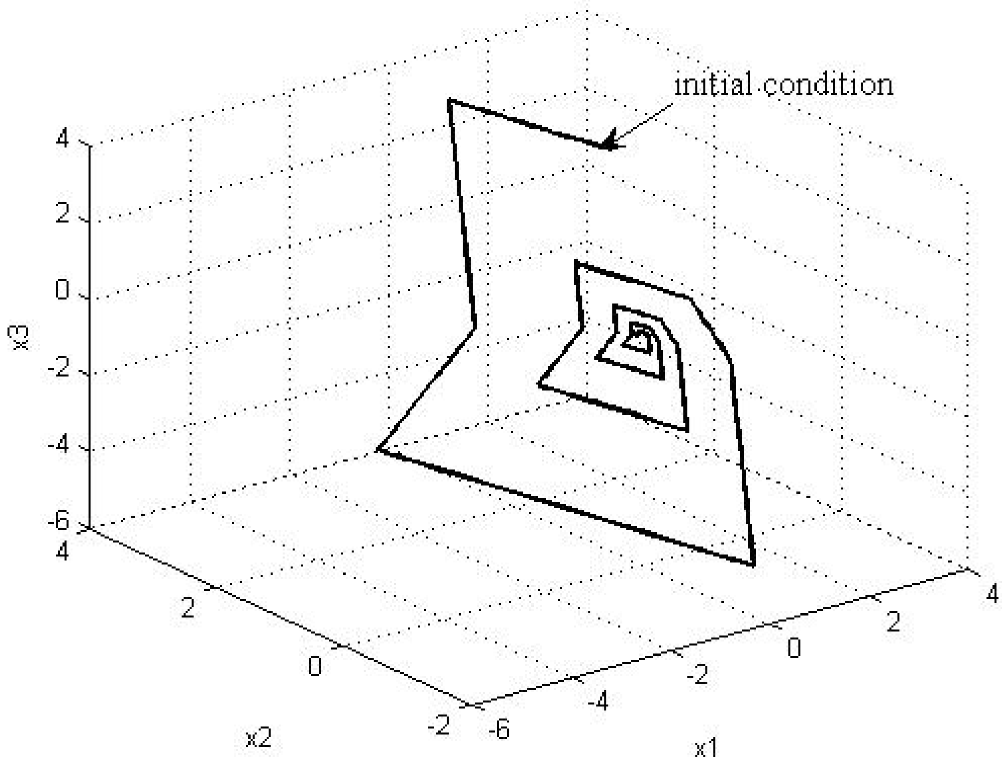


Figure 10a

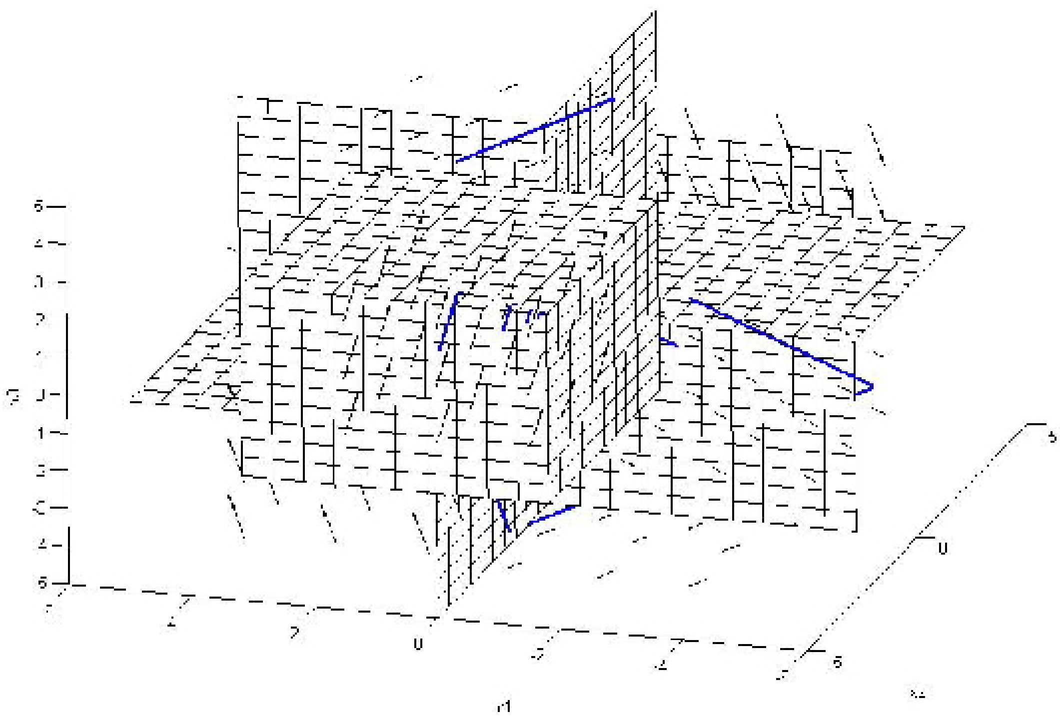


Figure 10b