Synchronization of switch dynamical systems

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Abstract

The well known synchronization method of Fujisaka and Yamada is adapted to a particular class of piece wise continuous dynamical systems. We give sufficient conditions for the underlying initial value problems, in order to define dynamical systems. For this purpose the Filippov regularization is used, the discontinuous initial value problem being switched into a differential inclusion. A generalized derivative for the considered class of functions is introduced.

0.1 Introduction

Differential equations with discontinuous right-hand side occur in many real problems and are widely used as simplified mathematical models of physical systems although the initial value problem (i.v.p.) need not have any classical solutions. Sometimes physical laws are expressed by discontinuous functions, for example a discontinuous dependence of the friction force on the velocity in the cases of dry friction, oscillating systems with combined dry and viscous damping, elasto-plasticity, electrical circuits, forced vibrations, brake processes with locking phase, control synthesis of uncertain systems etc. (see e.g. [Butenin *et al.*, 1987; Deimling, 1992; Popp & Stelter, 1990; Popp *et al.*, 1995; Rumpel, 1996] and their references).

In this paper we consider piecewise continuous dynamical systems, we called switch systems, modeled by the following autonomous i.v.p.

$$\begin{array}{l} \dot{x}(t) = f\,(x(t)) := \\ g(x(t)) + \sum_{i=1}^{n} \alpha_i \, sgn\,(x_i(t))\, e^i, \\ x(0) = x_0, \quad t \in I = [0, \infty), \quad \alpha_i \in \mathbf{R}, \end{array} \tag{1.1}$$

where f and g are vector-valued functions $f,\, g: \mathbf{R}^n \rightarrow \mathbf{R}^n$ (g being considered continuous with respect to state variable x), and e^i denotes the i-th canonical unit vector in $\mathbf{R}^n$.

The right-hand side of i.v.p. (1.1) is discontinuous with respect to the state variable x, due to the sign functions. Since the system is autonomous, we can assume, without loss of generality, that the initial condition is given at $t = 0$.

Our first goal is to find the assumptions on which the class of the i.v.p. (1.1) defines a dynamical system (d.s.). The second purpose is to explore the possibility to synchronize two such switch d.s. with chaotic motion. Using the computational aspects of the system behavior (the so called numerical dynamics), we find the maximum Lyapunov exponent, one of the powerful tool to diagnose whether or not the behavior is chaotic[1] (see [Lyapunov, 1907] or [Oseledec, 1968] for theory and [Benettin *et al.*, 1976; Eckmann & Ruelle, 1985; Parker & Chua, 1989; Wolf *et al.*, 1985] for numerical methods to find Lyapunov exponents for continuous d.s.). In purpose, a new concept of derivative for our class of functions f defined in (1.1) is introduced.

In [Kunze, 2000], the Lyapunov exponents for discontinuous i.v.p. are treated using the so called *cocylces*, instead of the linearization of i.v.p.,

[1]It was proved that for any definition of chaos, there may always be some "clearly" chaotic systems which do not fall under that definition (see [Brown & Chua, 1996]. However, we consider in this paper the definition given by Gulik [Gulik, 1992]: if the largest Lyapunov exponent is positive, then the behavior of the system is considered to be chaotic.

while in [Müller, 1995] the required linearized equations are supplemented by certain transition conditions at the discontinuities points.

One of the surprising fact about the chaotic systems (continuous or not) is that they can be synchronized. This refers to the tendency of two or more systems, which are coupled together, to undergo closely related motions. Synchronization between continuous chaotic systems has been the subject of many studies over the last few years. The most of these approaches uses the Lyapunov exponents or Lyapunov functions (see e.g. [Alligood *et al.*, 1997; Aswin *et al.*, 1996; Brown & Rulkov, 1997; Kapitaniak, 1993; Kapitaniak & Thylwe, 1996; Rumpel, 1996; Schuster *et al.* 1986; Fujisaka & Yamada, 1983; Wu & Chua, 1994]; in [Rilkov *et al.*, 2001] a numerical analysis study including double-valued functions is presented). In this paper the so called one-to-one coupling of Fujisaka and Yamada method [Fujisaka & Yamada, 1983] was adapted to synchronize two identical switch d.s. modeled by i.v.p. (1.1).

The structure of the paper is the following: in section 2 notions as generalized derivative, switch dynamical systems, are introduced; in section 3 differential inclusions, Filippov solutions, sufficient conditions to define switch d.s. and numerical integration of differential inclusions (namely the explicit Euler method) are presented; in section 4, the method to find the maximum Lyapunov exponent for continuous d.s. are adapted to the case of switch d.s. and in section 5 the synchronization theorem for the continuous case is used to synchronize switch d.s. The case of a generalized switch Chua's circuit is analyzed.

0.2 Notations and auxiliary results

Because the classical notion of derivatives at the discontinuity points of f cannot be used here, a new concept of derivative (which uses the classical derivative notion at the continuity points) is required.

If f is differentiable at some $x_0 \in \mathbf{R}^n$, the derivative f' will be given, as usual, by the Jacobi $n \times n$ matrix J.

Let D_i be open subsets of $\mathbf{R}^n$, for $i = 1, 2, \ldots, p$, such that $\mathbf{R}^n = \overline{\bigcup_{i=1}^{p} D_i}$ and let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a single-valued function.

Definition 2.1. Let f be differentiable on $\bigcup_{i=1}^{p} D_i$. We say that f is *generalized differentiable* at $x^* \in \mathbf{R}^n$ if the following limit exists and is finite

$$D\,f\,(x^*) := \lim_{x \to x^*} f\,'(x), \quad x \in \bigcup_{i=1}^{p} D_i\,. \tag{2.1}$$

Then, $D\,f\,(x^*)$ will be called the *generalized derivative* (*generalized Jacobi matrix* $\overline{J}\,(x^*)$) of f at x^*. We say that f is *generalized differentiable on* $\mathbf{R}^n$ if it is so at every $x^* \in \mathbf{R}^n$.

Notation: The class of functions f having generalized derivative on $\mathbf{R}^n$ will be denoted by $\mathcal{C}^1$.

The discontinuity set of f is contained in $M = \mathbf{R}^n \setminus \bigcup_{i=1}^{p} D_i$.
Example Let consider

$$f\,(x) = x + sgn(x)\,,$$

with $M = \{0\}$. $D_1 = \{x \in \mathbf{R}\,|\,\mathbf{x} < \mathbf{0}\}$, and $D_2 = \{x \in \mathbf{R}\,|\,\mathbf{x} > \mathbf{0}\}$. Then, for $x^* = 0$, we have $Df\,(0) = \lim\limits_{x \to 0} f\,'(x) = 1$.

It is easy to check the following proposition.

Proposition 2.1. Let consider the i.v.p. (1.1) with $g \in C^1[\mathbf{R}^n]$. Then $f \in C^1$ and

$$D\,f\,(x) = g\,'(x^*)\,, \quad x^* \in \mathbf{R}^n\,. \tag{2.2}$$

There are many mathematical definitions for d.s. (see [Schuster, 1989; Stuart & Humphries, 1996] and the references herein). We introduce the following definition, which uses the existence and optionally the uniqueness of solutions.

Definition 2.2. The i.v.p. (1.1) is said to define a *switch generalized d.s.* on $\mathbf{R}^n$ if for every $x_0 \in \mathbf{R}^n$ there exists a solution of i.v.p. (1.1) defined for almost all $t \in I$. If the solution is almost everywhere unique, then the i.v.p. is said to defines a *switch d.s.*

Remark. Many authors (see e.g. [Schuster, 1989] and the references therein) consider the concept of continuity of a d.s. as being with respect to the time variable. Hence, if time is a real variable ($t \in \mathbf{R}$), the system is called *continuous,* while if time is an integer variable ($t \in \mathbf{Z}$), the system is called *discrete*. Other authors consider the continuity (or Lipschitz continuity) with respect to the initial data (see e.g. [Stuart & Humphries, 1996]). However in the continuity case with respect to the state variable, for most of the standard assumptions leading to existence and uniqueness of the solutions, the continuous dependence on initial data follows.

In this paper we consider the state continuity concept.

If the right-hand side of an i.v.p. is a continuous function with respect to the time t and state variable x then, the i.v.p. may defines a d.s.

Example [Stuart & Humphries, 1996] The following continuous right-hand side i.v.p.

$$\dot{x} = \alpha\, x^3, \quad x(t_0) = x_0, \quad \alpha \in \mathbf{R},$$

has the local classical solution $x(t) = x_0/(1 - 2\,\alpha\, x_0^2\, t)^{1/2}$. For $\alpha \leq 0$ the i.v.p. defines a continuous d.s. on $\mathbf{R}$. For $\alpha > 0$ the equation does not defines a d.s. on any open set since the classical solution exists only for $t \in [\,0,\, 1/2\,\alpha\, x_0^2)$ and for $t = 1/2\,\alpha\, x_0^2$ it becomes unbounded.

If the right-hand side is discontinuous with respect t or / and x, the i.v.p. need not have classical solutions. One of the typical cases for discontinuous i.v.p. are modeled using the sign function.

Example [Filippov, 1988] Consider the discontinuous right-hand side equation

$$\dot{x} = 1 - 2\, sgn\,(x)\,,$$

with the classical solutions (Figure 1)

$$x(t) = \left\{ \begin{array}{l} 3t + C_1 \\ -t + C_2 \end{array} \right. , \quad C_1, C_2 \in \mathbf{R}\,,$$

As t increases, the classical solutions tend to the line $x = 0$, but it cannot be continued along this line, since the map $x(t) = 0$ so obtained, does not satisfy the equation in the usual sense (for it $\dot{x}(t) = 0$ and the right-hand side has the value $1 - 2\, sgn(0) = 1$). Hence there are no classical solutions of i.v.p. starting with $x(0) = 0$.

Therefore a generalization of the concept of solution is required.

0.3 Differential inclusions and switch dynamical systems

Let consider the discontinuous i.v.p.

$$\dot{x}(t) = f\,(x(t)), \quad x(0) = x_0, \quad t \in I\,, \tag{3.1}$$

with f a piece wise vector single-valued continuous function $f : \mathbf{R}^n \to \mathbf{R}^n$. In order to avoid the possible lack of solutions of i.v.p. (3.1), the problem may be restarted as a differential inclusion (d.i.) [Filippov, 1988]

$$
\begin{aligned}
&\dot{x}(t) \in F(x(t))\,, \\
&x(0) = x_0, \quad \text{for almost all } t \in I\,,
\end{aligned} \tag{3.2}
$$

where $F : \mathbf{R}^n \Longrightarrow \mathbf{R}^n$ is a vector set-valued map into the set of all subsets of $\mathbf{R}^n$, which can be defined in several ways. For the background of d.i. and set-valued functions we refer to [Aubin & Cellina, 1984; Aubin & Frankowska, 1990].

The simplest convex definition of F for our class of functions (defined by (1.1)) is obtained by the so called Filippov regularization (see [Filippov, 1988])

$$
F(x) = conv\,(f\,(x)), \quad x \in \mathbf{R}^n\,, \tag{3.3}
$$

where *conv* is the convex hull of f . The function F, given by (3.3) is a vector set-valued map into the set of all nonempty closed and convex subsets of $\mathbf{R}^n$. In the points x where the map f is continuous, $F(x)$ consists of one point which coincides with the value of f at this point (i.e. we get back $f\,(x)$ as right-hand side). In the discontinuity points, the set $F(x)$ is given by (3.3). Details and other regularization procedures can be found in [Filippov, 1988]. In (3.2) the key is the fact that we can ignore possible misbehavior of f on sets of null measure (the discontinuity set M) in the state space.

As example, the Filippov regularization of the usual sign map is the signum set-valued map Sgn (see Figure 2)

$$
Sgn\,(x) = \begin{cases} \{-1\}\,, & x < 0 \\ [-1, 1]\,, & x = 0 \\ \{+1\}\,, & x > 0 \end{cases} .
$$

Now we can give the concept of solution to (3.1) in terms of d.i. (3.2).

Definition 3.1. A (*Filippov*) *solution* of i.v.p. (3.1) is an absolutely continuous vector-valued map $x : I \to \mathbf{R}^n$ satisfying (3.2), almost everywhere on I .

The absolutely continuous functions are the weakest kind of solutions (see [Filippov, 1988] for properties of Filippov solutions). The background on existence and uniqueness of solutions to differential inclusions, can be found in [Filippov, 1988] or [Aubin & Cellina, 1984; Aubin, Frankowska, 1990].

Remark 3.1. Embedding f into a set-valued map F, which has enough regularity closely related to the trajectories of the original differential equation, we can stress the point that whenever f is continuous at x, then a solution to d.i. (3.2) satisfies the i.v.p. (3.1). Certainly, any classical solution to the i.v.p. (3.1) is a solution to the i.v.p. (3.2). Hence we are justified to call a solution of i.v.p. (3.1) as a solution of the i.v.p. (3.2) (see [Filippov, 1988]).

The so called *Péano functions*, (functions upper semicontinuous with non-empty closed and convex values) verify the assumptions in the Péano's existence theorem for differential inclusions (see [Aubin, Frankowska, 1990]).

Definition 3.2. F satisfies a *growth condition* (g.c.) on $\mathbf{R}^n$ if there exist constants $K_1, K_2 \geq 0$ with

$$\| \xi \| \leq K_1 \| x \| + K_2 \,,$$

for all $\xi \in F(x)$, $x \in \mathbf{R}^n$.

The g.c. implies that all solutions remain in some bounded subset and it is used instead of global boundedness of the right-hand side (compare [Aubin & Cellina, 1984; Lempio, 1995; Taubert, 1981]). As example the Sgn set-valued functions satisfy the g.c.

Using the Filippov regularization, the obtained set-valued map F belongs to Péano's class.

Let apply now the Filippov regularization to i.v.p. (1.1). One obtain

$$\begin{array}{l} \dot{x}(t) \in F(x(t)) := g(x(t)) + \sum_{i=1}^{n} \alpha_i \, Sgn\,(x_i(t))\, e^i, \\ x(0) = x_0, \quad \text{for almost all } t \in I\,. \end{array} \tag{3.4}$$

Remark 3.2. On mild assumptions, a d.i. (as (3.2) or (3.4)) has a Filippov solution that happens to be even unique, but it could have multiple solutions too.

There are several sufficient conditions to assure the uniqueness e.g. *one sided Lipschitz conditions* [Filippov, 1964; Lempio, 1990; Lempio, 1995] (i.e. we have $(y' - y'', x' - x'') \leq \lambda \| x' - x'' \|^2$, uniformly in t and for all $y' \in F(x')$, $y'' \in F(x'')$, with $x',\ x'' \in \mathbf{R}^n$). The function $-Sgn$ verifies one sided Lipschitz condition. Other uniqueness (and existence too) condition is the maximal monotonicity (see [Aubin & Cellina, 1984; Aubin & Frankowska, 1990]). A general criterion for nonuniqueness does not exists. However for the i.v.p. (1.1) the positiveness of some α_k in (3.4) seems to be adequate for nonuniqueness (see [Danca 2001 a]).

Example Let the discontinuous i.v.p. $\dot{x} = sgn\,(x)$, $x(0) = 0$. There is no classical solution starting from 0. However considering the corresponding d.i. $\dot{x} \in F(x) = Sgn\,(x)$, for almost all $t \in I$, there are multiple Filippov solutions: $x(t) = 0$ for $t \leq t^*$ and $x(t) = \pm(t - t^*)$ for $t > t^*$, where $t^* \geq 0$ could be ∞.

Example If we consider the i.v.p. $\dot{x} = -sgn\,(x)$, $x(0) = 0$, then there is a unique Filippov solution for the d.i. $\dot{x} \in -Sgn\,(x)$

$$x(t) = \begin{cases} t^* - t\,, & t \leq t^* \\ 0\,, & t > t^*\,, \end{cases}$$

and the trajectory can be continuously extended from $x = 0$ for $t > t^*$.

The following theorem is the main result of this section, and states the conditions in which the i.v.p. (1.1) defines a switch (generalized) d.s. using the underlying i.v.p. (3.4). The proof can be found in [Danca, 2001 a].

Theorem 3.1. *Let the i.v.p. (1.1) with g Lipschitz continuous and satisfying a g.c. Then the i.v.p. (1.1) defines a generalized switch d.s. If moreover, $\alpha_i < 0$ for all i, then the i.v.p. (1.1) defines a switch d.s.*

Sketch of proof. If g verify a g.c. then F verifies a g.c. too. Then, it can be proved that the i.v.p. (1.1) has Filippov solutions on some interval $[0, T]$ (via. i.v.p. (3.4), see Remark 3.1) and the i.v.p. defines a switch generalized d.s. If moreover $\alpha_i < 0$, g being Lipschitz continuous, it can be proved that the solution is unique (see Remark 3.2), and using Definition 2.2, the i.v.p. defines a switch d.s. In order to extend the existence interval to $[0, \infty)$, necessary to define a d.s. a compactness condition is needed (see [Aubin & Cellina, 1990, pp. 101]). In our case, is easy to see that $F(x)$ is a compact set.

The d.i. (3.4) is used in Theorem 3.1 only as a tool to prove the existence/uniqueness of the Filippov solutions.

Remark Another way to approach switch d.s. modeled by i.v.p. (1.1) is the use of the Cellina's Theorem [Aubin & Cellina, 1990, Theorem 1, pp. 84 and Aubin & Frankowska, 1990, Theorem 9.2.1, pp. 358] to approximate the discontinuous i.v.p. with a continuous one (see also [Danca, 2001 b]).

Test problem. Let consider the following discontinuous problem modeling a Chua circuit ([Brown, 1993; Chua *et al.*, 1993])

$$\begin{aligned} \dot{x}_1 &= -\alpha\,(b+1)\,[x_1 - k\ sgn\,(x_1)] + \alpha\, x_2 \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -\beta\, x_2\,, \end{aligned} \tag{3.5}$$

Using the parameters indicated in [Brown, 1993]: $-\alpha(b+1) = -2.57$, $\alpha = 9$, $\beta = 15.7$ and $k = 1.5$, we have

$$\begin{aligned} f \quad &: \quad \mathbf{R}^3 \to \mathbf{R}^3, \quad f\,(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} \\ &= \begin{pmatrix} -2.57\,x_1 + 9\,x_2 + 3.86\,sgn\,(x_1) \\ x_1 - x_2 + x_3 \\ -15.7\,x_2 \end{pmatrix}. \end{aligned}$$

The discontinuity surface S is defined by the equation $x_1 = 0$. The open subsets D_i (see Definition 2.1) are $D_1 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, x_1 < 0\}$, and $D_2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, x_1 > 0\}$. The i.v.p. has no global classical solution on $[0, \infty)$. The Filippov regularization gives us the following d.i.

$$\begin{aligned} &\dot{x}_1 \in -2.57\,x_1 + 9\,x_2 + 3.86\,Sgn\,(x_1) \\ &\dot{x}_2 = x_1 - x_2 + x_3 \\ &\dot{x}_3 = -15.7\,x_2\,. \end{aligned} \tag{3.6}$$

with

$$F(x) = \begin{pmatrix} -2.57\,x_1 + 9\,x_2 \\ x_1 - x_2 + x_3 \\ -15.7\,x_2 \end{pmatrix} + 3.86\,Sgn\,(x_1)\,e^1.$$

The i.v.p. (3.5) defines a generalized switch d.s. because the assumptions in Theorem 3.1 can be reasonably checked, the map $g(x) = (-2.57\,x_1 + 9\,x_2\,,\ x_1 - x_2 + x_3,\ -15.7\,x_2)^T$ being a linear one. The solution is not unique due to the presence of $+Sgn\,(x_1)$ (see Remark 3.2). In Figure 3, f_1 and $F_1(x_1, x_2) = -2.57\,x_1 + 9\,x_2 + 3.86\,Sgn\,(x_1)$, are plotted.

The simplest numerical method for d.i. is the explicit Euler method. It is known that under some assumptions any sequence of piecewise linear interpolations of some discrete trajectories has a convergent subsequence in $C[0, T]$, to some trajectory of a d.i. [Dontchev & Lempio 1992; Lempio 1990; Lempio 1995]. Let N be a natural number $N \in \mathbf{N}' \subset \mathbf{N}$, $\mathbf{N}'$ denoting a subsequence of N tending to infinity, $h = T/N$, and an equidistant grid

$$t_0 < t_1 < t_2 < \ldots < t_N = T\,.$$

For each h let be a set of linear continuous functions approximating the whole solution set X_h of (3.2). We associate with (3.2) a sequence of discrete-time inclusions in the form

$$\begin{aligned} &y_{k+1} \in G_k^N(h; y_k), \\ &k = 0, 1, \ldots, N-1, \quad y_0 = x_0, \end{aligned} \tag{3.7}$$

where $G_k^N : \mathbf{R}^n \Longrightarrow \mathbf{R}^n$ is a discrete-time set-valued map. A solution of (3.7) is any sequence of $N+1$ vectors $y_0, y_1, \ldots, y_N$ that satisfies (3.7) for $k = 0, 1, \ldots, N-1$. The main problem is to define a family of mappings G_k^N such that the solutions of the problem (3.7) suitable approximate in some sense the set of solutions X_h. In order to define a link between the trajectories of the continuous-time i.v.p. (3.2) and discrete-time (3.7), we associate the state y_k of the discrete-time system with the moment t_k in the continuous time-scale, i.e. we compare y_k with $x(t_k)$.

The explicit (forward) Euler method for solving a d.i. is the set-valued version of the classical discretization method for differential equation with

$$G_k^N(h; y_k) = y_k + h\, F(t_k, y_k) \tag{3.8}$$

The Euler convergence theorem for d.i. is presented in many works, and various forms (see e.g. [Filippov, 1988, Theorem 1, pp.77; Aubin & Cellina, 1984, Lemma 1, pp. 99; Aubin & Frankowska, Theorem 10.1.3, pp. 390], or the papers [Lempio, 1998; Lempio, 1995; Lempio, 1990; Taubert, 1981]). Its constructive proof (for example Euler broken lines) uses the idea of the classical Péano theorem to prove existence of solutions to d.i.

Theorem 3.2. (Péano's Theorem, [Dontchev & Lempio, 1992]). Let the i.v.p. (3.2) with F a Péano function satisfying a g.c. Then, every sequence $(y^N)_{N \in \mathbf{N}'}$, defined by (3.7-3.8), with $y^N \in X_h$ for $N \in \mathbf{N}'$, has a subsequence which converges as $N \to \infty$ uniformly in I, to some solution of (3.2).

Because, generally, the solution of the inclusion (3.7) is not unique, the question is how to reasonably choose y_{k+1} of $G_k^N(h; y_k)$ at each step of the discrete system. y_{k+1} would be selected randomly (as in the present paper) or by a suitable criterion (see [Dontchev & Lempio, 1992] and [Kastner-Maresch & Lempio, 1993] for selection strategy).

If the solution is unique the whole sequence of approximations converges to this solution.

Test problem. Let consider the Chua's circuit (3.5). For the behavior is chaotic, this motion being deduced from the bifurcation diagram (Figure 4), where the maximum values of versus the control parameter β was plotted. The corresponding trajectory was obtained using the explicit Euler method (Figure 5).

In Figure 6 a computer graphic simulation using Matlab was plotted.

The solution is not unique (see Remark 3.2). This could be interpreted here in the following manner: the value of $\dot{x}_1$, for $x_1 = 0$, is uncertain and can take any value in the range $9x_2 + [-3.86, 3.86]$. Hence, any numerical solution corresponding to $\dot{x}_1$ in this range, can be considered as a possible motion of system for a period of time.

Remark 3.3. In the case of the explicit methods (as the forward Euler method used in this paper), the trajectory could have corners on some time subintervals (Figure 7), where the exact solution does not exists in the classical sense but only in the Filippov sense and crosses several time the discontinuity surface S (see also [Danca, 2001 b]). Examples can be found in [Wiercigroch & de Kraker, 2000]. To avoid the nonsmoothness of the solutions of a differential inclusion, highly consistent implicit methods with additional procedures can be used, as the implicit Runge-Kutta methods (compare [Dontchev & Lempio, 1992]).

0.4 Lyapunov exponents

Lyapunov exponents are a generalization of the eigenvalues at an equilibrium point. It quantifies the average growth of infinitesimally small errors in the initial point . It is well known that if the largest Lyapunov exponent of a d.s. is positive, then two trajectories starting close to one another in the phase space, will move exponentially away from each another for small times on the average. The existence of at lest one positive Lyapunov exponent is often used as definition of chaos (the Gulik's definition [Gulik, 1992]). However, one should not rely solely on this technique to certify a motion to be chaotic. Other tests (spectral analysis, Poincaré functions, bifurcation diagrams or fractal dimension) should also be used to confirm the presence of chaos.

The existence of the Lyapunov exponents is assured by the criterion provided by the famous *multiplicative ergodic theorem* of Oseledec [Oseledec, 1968] that relies on some ergodic probability measure invariant with respect to the flow. The Oseledec's theorem implies that the Lyapunov exponents of a function f exist in great generality if f is a $C^1[\mathbf{R}^n]$ function and the Jacobi matrix is Hölder continuous for some exponent θ (see [Guchenheimer & Holmes, 1983]).

There are two general methods to calculate the Lyapunov exponents. One is for data generated by a continuous i.v.p. (see e.g. [Schuster, 1989]), and the other for experimental time series data (see e.g. [Eckmann & Ruelle, 1985; Wolf *et al.*, 1985]). The last method does not requires a priori knowledge of the system equations. The procedures to compute Lyapunov exponents can be found also in [Benettin *et al.*,1976; Benettin *et al.*1980; Wolf *et al.*, 1985].

Consider the continuous i.v.p.

$$\dot{x}(t) = f\,(x(t)), \quad x(0) = x_0, \quad t \in [\,0, \infty) \,. \tag{4.1}$$

with $f : \mathbf{R}^n \to \mathbf{R}^n$ a continuous vector-valued function. Let a trajectory in $\mathbf{R}^n$, called "fiduciary" [Wolf *et al.*, 1985], starting from x_0 and a nearby trajectory starting from $x_0 + \varepsilon(0)$, ε being the distance function between the two trajectories. Then at small later time this distance becomes $\varepsilon(t)$ (see Figure 8 (a)).

The time evolution for ε is found from the variational equations (see e.g. [Parker & Chua, 1989])

$$\dot{\varepsilon}(t) = J\,(x(t))\varepsilon(t)\,, \tag{4.2}$$

where J is the Jacobi matrix evaluated at the initial value $x(t)$, being in general time-dependent even if the i.v.p. is autonomous. The initial conditions are taken in general $\varepsilon(t_0) = I$.

Let the assumptions in Oseledec's theorem hold. Then the following limit exists and defines the Lyapunov exponents

$$\lambda_i = \lim_{t\to\infty} \left(\frac{1}{t} \log \mid \sigma_i \mid \right), \quad i = 1, 2, \ldots, n\,, \tag{4.3}$$

where σ_i are the eigenvalues of J ,for x_0 ranging over $\mathbf{R}^n$. *lim* can be replaced by *lim sup* to guarantee the existence of the Lyapunov exponents.

Remark. It is easy to see that if $x_0 = x^*$, where x^* is an equilibrium point, then the Lyapunov exponents are equal to the real parts of the eigenvalues of $J\,(x^*)$ (see e.g. [Parker & Chua, 1989]). If $x_0 \neq x^*$, the trajectory starting from x_0 tends to x^* for $t \to \infty$, i.e. lies in the basin of attraction of the equilibrium point. Then, since the Lyapunov exponents are defined in the limit as $t \to \infty$, any transient can be ignored and, the Lyapunov exponents of x^* and x_0 are the same.

In order to make a numerical estimation of ε we need first to integrate (4.1), to find $x(t)$. It can be prove (see e.g. [Schuster, 1989]) that at the time t the largest Lyapunov exponent, λ_m, is given as follows

$$\parallel \varepsilon(t) \parallel \approx e^{\lambda_m t}.$$

To avoid the possible overflow in the computer, one calculate the divergence of nearby trajectories for finite step-size and then one renormalize $\varepsilon(i\tau)$, $i = 1, 2, \ldots, n$ to unity after each step τ and one take the average (see [Benettin *et al.*, 1976; Benettin *et al.*, 1980; Eckmann & Ruelle, 1985; Parker & Chua, 1989]). Here τ must be not necessarily equal to the integration step. Finally we have (see Figure 8 (b))

$$\lambda_m = \lim_{n\to\infty} \frac{1}{n\,\tau} \sum_{i=1}^{n} \ln \|\, \varepsilon(\,\tau\, i)\| \,,$$

where

$$\varepsilon(\tau) = \varepsilon(0) e^{\lambda_1 \tau}, \ \varepsilon(2\tau) = \frac{\varepsilon(\tau)}{\| \, \varepsilon(\tau) \, \|} e^{\lambda_2 \tau}, \ \ldots ,$$

Consider now the discontinuous i.v.p. (1.1) with $g \in C^1[\mathbf{R}^n]$. The Jacobi matrix J is not defined at $x \in M$, but, using Definition 2.1 and Proposition 2.1, we can use the generalized Jacobi matrix, $\overline{J}\,(x) = \mathcal{D}\, f\,(x)$, given by (2.1) and defined at all the points $x \in \mathbf{R}^n$.

Hence, the above steps can be used in order to find λ_m.

Remark 4.1. The equation (4.2) will be considered only as an approximated model for the deviation ε in the neighborhood of the discontinuity surface. Suppose that the two analyzed trajectories (Figure 8) approach the discontinuity surface. Then, ε could have different evolutions before and after crossing (Figure 9). Hence, the behavior could remains chaotic (Figure 9 (a)) or could becomes regular (Figure 9 (b)). The problem could be avoid using numerical method with a high consistency order (See Remark 3.3). Our numerical experiments allowed us to the conclusion that, for our class of switch systems, the chaotic behavior, existing before the crossing, still persists after the crossing.

In order to find the Lyapunov exponents (using the algorithm presented in Section 3.1), we need to solve numerically the systems (4.1) and (4.2) with $\overline{J}$ instead J.

Test problem Using the above algorithm we found that for the Chua's generalized discontinuous d.s. (3.5) the largest Lyapunov exponent for $\beta = 15.7$ is $\lambda_m \approx 0.39$. It is interesting to see that the largest Lyapunov exponent for the underlying continuous Chua's circuit is $\lambda_m \approx 0.48$.

0.5 Synchronizing dynamical systems

Consider first, a chaotic continuous d.s. $\dot{x}(t) = f\,(x(t))$ with $n \geq 3$. Let Λ be a p - dimensional chaotic attractor in a m - dimensional phase space $(\,m > p)$.

Definition 5.1. The set Λ is a *Milnor attractor* [Milnor, 1985], if it is Lyapunov stable (i.e. the basin of attraction, $\beta(\Lambda)$, has a positive Lebesgue measure);

Λ is an asymptotically stable attractor if it is a Milnor attractor, and $\beta(\Lambda)$ is a neighborhood of Λ.

However, because it can happen, that $\beta(\Lambda)$ do not include the neighborhood of the attractor, the weaker Milnor attractor notion is usually used (see [Milnor, 1985] for the background of related notions).

If two d.s. evolve on an asymptotically stable n-dimensional attractor Λ in $2n$- dimensional phase space $\mathbf{R}^{2n}$, given by the relation $x = y$, we can locally synchronize them using the one-to-one coupling

$$\begin{aligned} &\dot{x}(t) = f\,(x(t)) + c(y(t) - x(t)), \\ &\dot{y}(t) = f\,(y(t)) + c(x(t) - y(t)), \\ &x(0) = x_0, \;\; y(0) = y_0, \;\;\; x_0 \neq y_0, \end{aligned} \tag{5.1}$$

for some positive range of the coupling coefficient c, i.e. the synchronized (or synchronous) state $z(t) = x(t) - y(t)$ becomes stable. Synchronization can be *global*, when the equilibrium $z = 0$ is asymptotically stable, meaning that no matter what initial conditions x_0, y_0 are taken, the systems will synchronize, or *local* when the equilibrium is only stable. If $c = 0$ then there is no synchronization. The synchronization state becomes stable equilibrium if the following well known theorem of Fujisaka and Yamada holds [Fujisaka & Yamada, 1983].

Theorem 5.1. Let λ_m be the largest Lyapunov exponent of the continuous dynamical system $\dot{x}(t) = f\,(x(t))$. Assume one-to-one coupling (5.1). If $c > \lambda_m/2$ then the coupled systems satisfies local synchronization. That is the synchronization state $z(t) = x(t) - y(t) = 0$ is a stable equilibrium.

P r o o f : The variational equations for the synchronized trajectory of (5.1) are, for $x, y \in \Lambda$

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} A(t) - cI & cI \\ cI & A(t) - cI \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{5.2}$$

where A is the Jacobi matrix of f evaluated along the synchronized trajectory $x(t) = y(t)$. Subtracting in (5.1) one obtain

$$\dot{B}(t) = (A(t) - 2cI)B(t), \tag{5.3}$$

where B is the Jacobi matrix of the flow z.

Now, the linearization in the neighborhood of the attractor Λ allows us to reduce the problem of stability of the attractor to the problem of stability of the fixed point $z(t) = 0$ of the equation (5.3), based on the fundamental results of the linear stability. The spectrum of Lyapunov exponents of equation (5.3), given by the characteristic equation, can be divided into two subsets: $\lambda^1 = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ associated with the evolution on the invariant manifold $x = y$, with at least one of Lyapunov exponents always positive, and $\lambda^2 = \{\lambda_1 - 2c, \lambda_2 - 2c, \ldots, \lambda_n - 2c\}$ describing the evolution transverse to above manifold. Next, if the largest Lyapunov exponent λ_m verifies $c > \lambda_m/2$, then obviously the chaotic attractor Λ is stable in $\mathbf{R}^{2n}$. If $c < \lambda_m/2$ the manifold $x = y$ is a repeller, the synchronization being not possible .

A rigorous theory of the transverse dynamics near an invariant submanifold, as Λ, can be found in [Ashwin *et al.*, 1994; Ashwin *et al.*, 1996].

Consider now the one-to-one coupling of two identical chaotic switch d.s. (1.1) with $g \in C^1[\mathbf{R}^n]$ Then, using the generalized Jacobi matrix, the Synchronization Theorem 5.1 can be applied (see Proposition 2.1).

Remark. The assumption $g \in C^1[\mathbf{R}^n]$ assures, besides the possibility to find the Lyapunov exponents and synchronization, the sufficient conditions for i.v.p. (1.1) to define a switch d.s.

Application. Let us consider again the Chua generalized switch d.s. (3.5). Using the one-to-one synchronization algorithm, we obtained the result illustrated in Figure 10. The synchronization algorithm was applied beginning from $t = t_0$. Before this value, c is taken 0, and the two trajectories are separated. After $t = t_0$ if we chose $c > \lambda_m/2 \approx 0.19$, after small time, the systems becomes synchronized.

Remark.

i) In [Stefanski & Kapitaniak, 2000] the chaos synchronization is used to estimate the largest Lyapunov exponent for continuous d.s. Hence, making a bifurcation diagram of the state z versus c, the searched value λ_m is twice the smallest value of the coupling coefficient c for which the synchronization takes place (i.e. z vanishes). Obviously, this method can be used to find λ_m for switch d.s. too.

ii) The value choose for c, $c = 0.30$, is sensible larger than $\lambda_m/2 \approx 0.19$, (probably) due to the influence of the discontinuity (see Remark 4.1).

Conclusions Although the class of i.v.p. (1.1), has a particular form, it can be founded in many practical problems.

We introduced Definition 2.2 because not all discontinuous i.v.p. defines a d.s. Hence, in order to find the assumptions in which the i.v.p. (1.1) defines a d.s., the function g must be of $C^1[\mathbf{R}^n]$ class. Introducing the concept of generalized derivative, it seems that is possible to find the Lyapunov exponents and synchronize two such identical d.s. having chaotic motion. We realized the synchronization of two generalized Chua circuits modeled by a discontinuous i.v.p.

The synchronization proposed in this paper can be applied to nonautonomous switch d.s. too.

In [Filippov, 1988] several aspects of the general class of problems (3.1) are treated.

Some problems which deserve future investigations are a qualitative (or quantitative) study of the system behavior in the discontinuity points and a generalization of our derivative concept to a larger class of i.v.p. (1.1).

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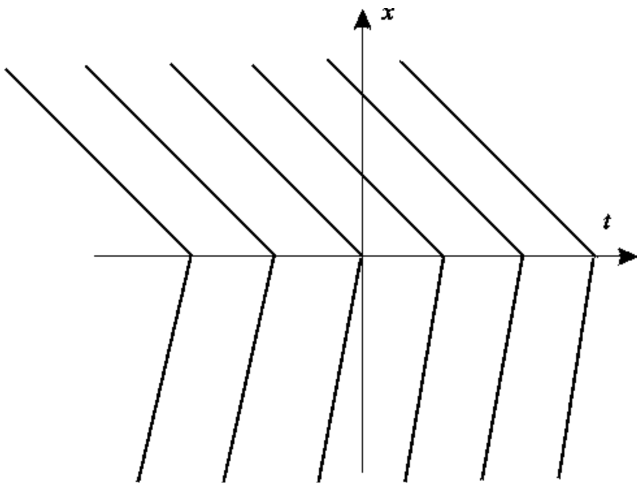


Fig. 1. Solutions of the equation
$\dot{x}=1-2sgn(x)$, for $x \neq 0$.

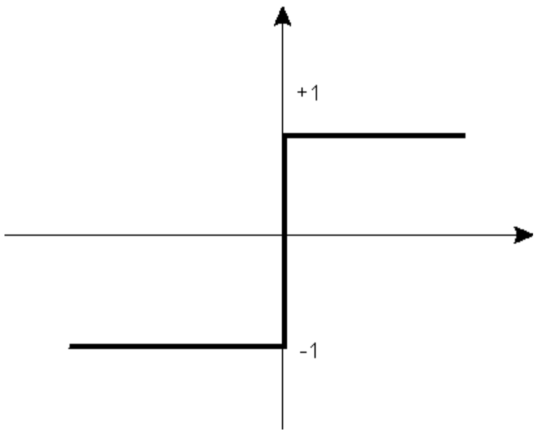
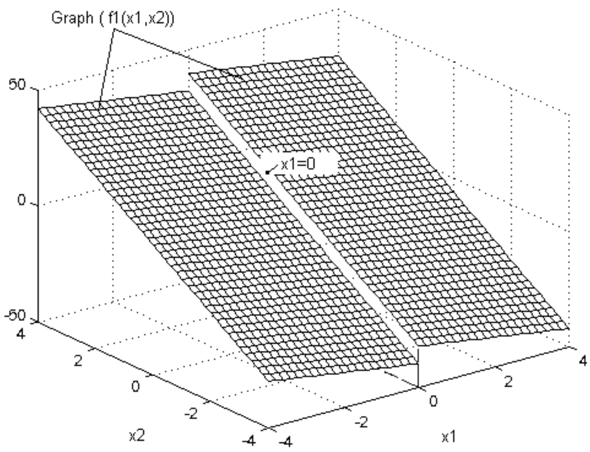


Fig. 2. The graph of the set-valued function Sgn(x).



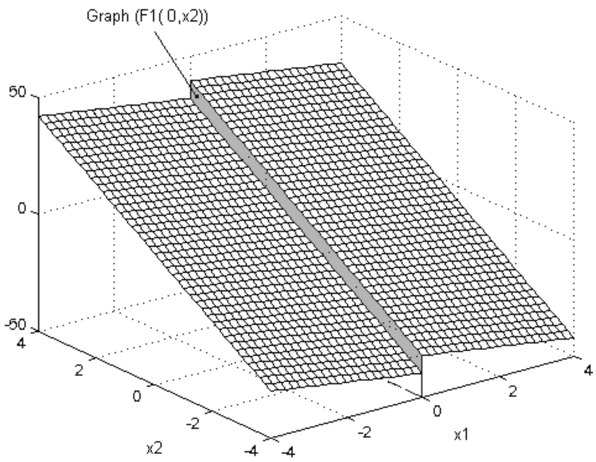


Fig. 3. (a) The graph of the first component, f1, of the right-hand side of (8); (b) The graph of the corresponding set-valued

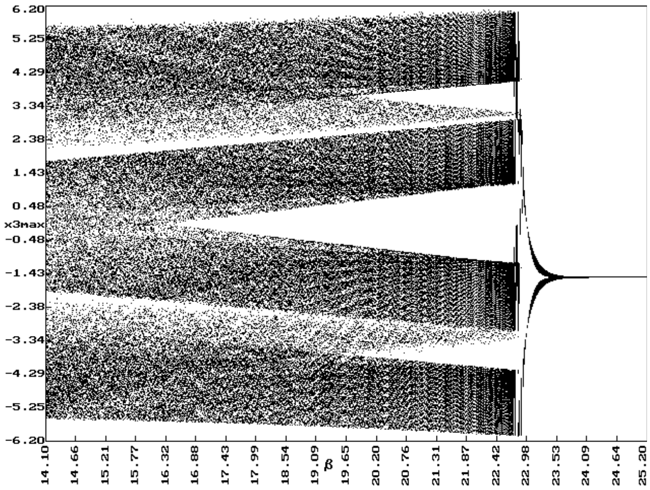
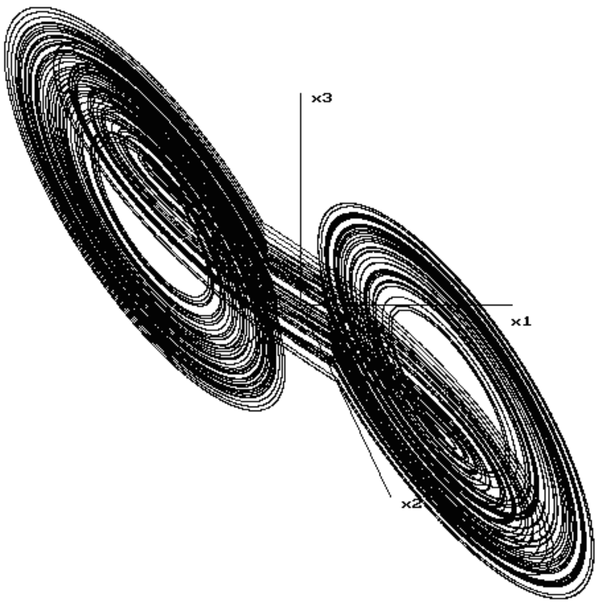


Fig. 4. Bifurcation diagram of the component x3 max of the dynamical system (8) versus the control parameter β.



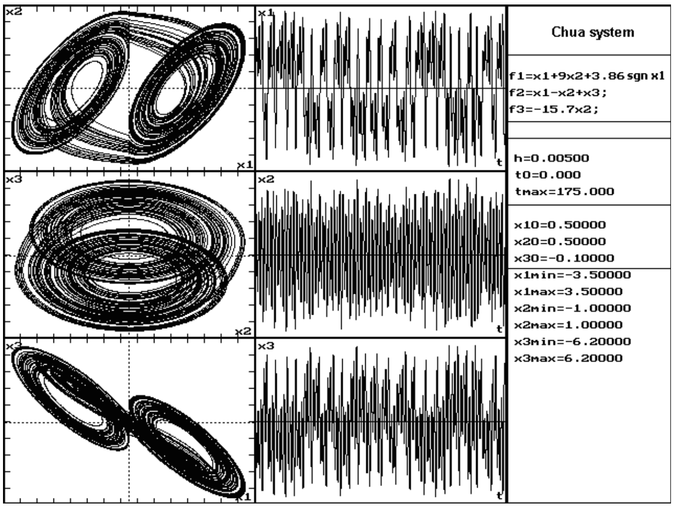


Fig. 5. A chaotic trajectory of the switch Chua circuit (8) obtained with the explicit Euler method. (a) Phase portraits and

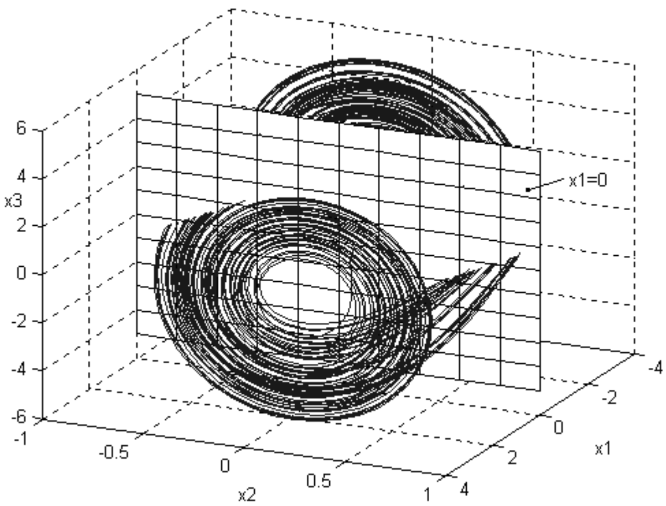


Fig. 6. Three-dimensional view of a chaotic trajectory and

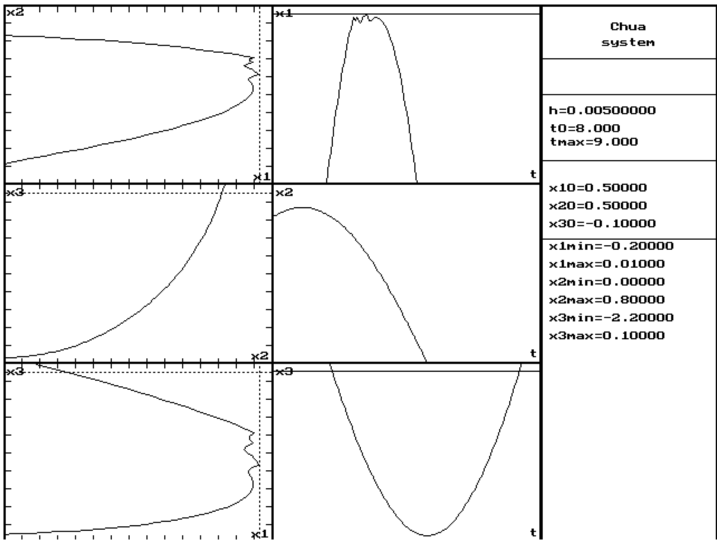


Fig. 7. Detail of a chaotic trajectory. Due to x1 discontinuity, only this component presents corners near the discontinuity surface $x_1 = 0$.

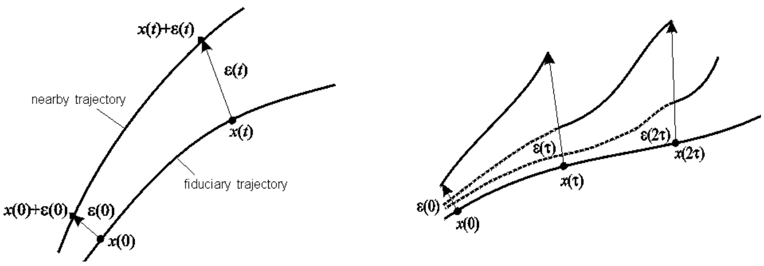


Fig. 8. (a) Exponential separation of two closed trajectories (schematically); (b) Renormalization of errors along a trajectory

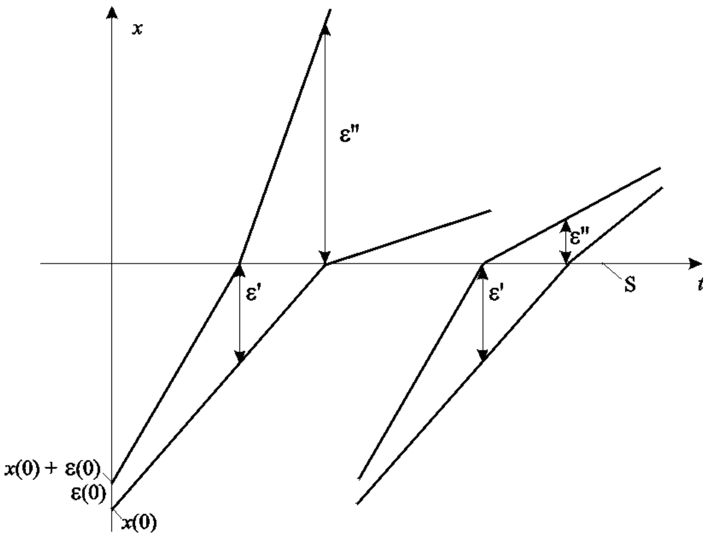


Fig. 9. The divergence of two closed trajectories in the phase space $[0, \infty) \times R$ near the discontinuity surface $S, x = 0$. (a) The ■ distance between the trajectories still increases after the surface crossing: the system behavior could remain chaotic; (b) The distance decreases, and the motion could be not chaotic after the crossing (schematically).

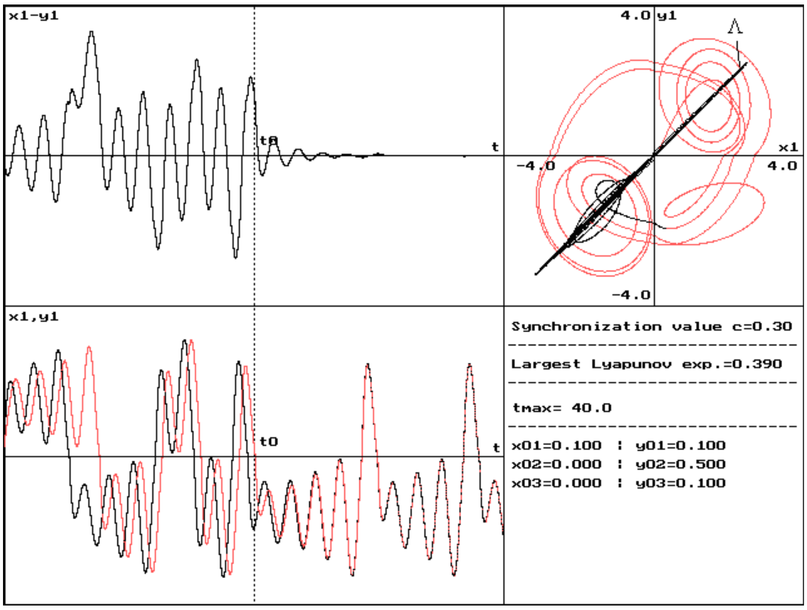


Fig. 10. Synchronization of two identical switch Chua circuits modeled by (8).