Lyapunov exponents of a class of piecewise continuous systems of fractional order

Marius-F. Danca

Abstract In this paper, we prove that a class of piecewise continuous autonomous systems of fractional order has well-defined Lyapunov exponents. To do so, based on some known results from differential inclusions of integer order and fractional order, as well as differential equations with discontinuous right-hand sides, the corresponding discontinuous initial value problem is approximated by a continuous one with fractional order. Then, the Lyapunov exponents are numerically determined using, for example, Wolf's algorithm. Three examples of piecewise continuous chaotic systems of fractional order are simulated and analyzed: Sprott's system, Chen's system and Simizu-Morioka's system.

Keywords Piecewise continuous function · Fractional-order system · Piecewise continuous system of fractional order · Lyapunov exponent · Chaotic system

1 Introduction

Nowadays discontinuous systems of fractional order stand out as a novel topic of broad interest since they provide a logical link between the fractional derivative approach to descriptive systems and their physical properties, such as dry friction, forced vibration, brake processes with locking phases, as well as stick and slip phenomena.

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However, to the best of our knowledge, there are yet very few works and results on discontinuous systems of fractional order. Also, although most dedicated numerical methods for differential equations of fractional order can be used to integrate abruptly discontinuous equations of fractional order systems, they did not come with rigorous mathematical justification (discontinuous equations may not even have classical solutions).

In this context, defining and calculating Lyapunov exponents (LEs, called by Oseledec "Liapunov characteristic numbers" [1] or "characteristic exponents" by Eckmann and Ruelle [2]) of systems modeled by fractional-order differential equations (FDEs) with discontinuous righthand sides, post a real challenge (see e.g. [3] pp. 237–231, [4,5] and [6–9] and the references there, on calculating LEs in PWC and FDE systems respectively).

In this paper, the existence of LEs of piecewise continuous systems of fractional order is proved.

The studied systems are modeled by the following Caputo-type autonomous piecewise continuous (PWC) Initial Value Problem (IVP)

$$D_*^q x = f(x) := g(x) + A(x)s(x), \quad x(0) = x_0, \quad t \in [0, T], \tag{1}$$

where $T > 0$, $g : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear, at least continuous function, $s : \mathbb{R}^n \to \mathbb{R}^n$, $s(x) = (s_1(x_1), s_2(x_2), ..., s_n(x_n))^T$ a piecewise function, with $s_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, ..., n$, piecewise constant functions ($sign$ or Heaviside functions in many applications), $A \in \mathbb{R}^{n \times n}$ a square matrix of real functions, and D_*^q, with q being some positive real number, stands for the Caputo fractional derivative.

The discontinuity of f appears if the following assumption is considered:

(H1) At least one element of $A(x)s(x)$ is discontinuous.

As in most of practical examples, $s_i(x_i) = sign(x_i)$. For example, usual case of $n = 3$ and $s(x) = (sign(x_1), sign(x_2), sign(x_3))^T$, one example is the fractional-order PWC Sprott's system [10,11]

$$
\begin{aligned}
D_*^{q_1} x_1 &= x_2, \\
D_*^{q_2} x_2 &= x_3, \\
D_*^{q_3} x_3 &= -x_1 - x_2 - ax_3 + bsign(x_1),
\end{aligned}
\tag{2}
$$

with $a = 0.5$, $b = 1$, where

$$
g(x) = \begin{pmatrix} x_2 \\ x_3 \\ -x_1 - x_2 - ax_3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{pmatrix}.
$$

Replacing the PWC functions, such as *sign* or Heaviside function, with continuous functions, represents an usual setting in many works (see e.g. [12]). Using some known results of differential inclusions, and following the way presented in [13], we show why and how this approximation can be done.

More precisely, we prove that systems modeled by the IVP (1) can be approximated with continuous systems of fractional order, for which the variational equations defining LEs are well defined.

The paper is organized as follows: Section I presents notations and preliminaries to be utilized, Section II presents the way in which the IVP (1) can be continuously approximated and Section III deals with the variational equations which define LEs. In Section IV, the LEs for three examples of PWC chaotic systems of fractional order are determined. The Conclusion Section ends this paper.

2 Preliminaries

Notation 1 *Denote by $\mathcal{M}$ the null discontinuity set of f, generated at the discontinuity points of s_i, which are defined in the IVP (1).*

$\mathcal{M}$ has zero Lebesgue measure, $\mu(\mathcal{M}) = 0$, and divides $\mathbb{R}^n$ into several ($m > 1$) open disjunct and connected sub-domains, $\mathcal{D}_i \subset \mathbb{R}^n$, $i = 1, 2, ..., m$, such that $\mathbb{R}^n = \bigcup_{i=1}^m \overline{\mathcal{D}_i}$. The discontinuity points belong to the union of the boundaries of $\mathcal{D}_i$, i.e. $\mathcal{M} = \bigcup_{i=1}^m bndry(\mathcal{D}_i)$ (see e.g. [14] for a tutorial on discontinuous dynamical systems).

Definition 2 A function $f : \mathbb{R}^n \to \mathbb{R}^n$, is called *piecewise continuous* if it is continuous on $R^n \setminus \mathcal{M}$ and has finite (possible different) limits on $\mathcal{M}$.

Under Assumption **H1**, f, defined by (1), is PWC. The following assumption on g will be imposed:

(H2) g is differentiable on $\mathbb{R}^n$.

Because the PWC functions, s_i, are linear on $\mathcal{D}_k$, $k = 1, 2, ..., m$, they are differentiable on $\mathcal{D}_k$. Therefore, the following property holds

Property 1 f is PWC on $\mathbb{R}^n$ and differentiable on $\mathcal{D}_k$, $k = 1, 2, ..., m$.

The differentiability of g is required for LE. [1]
For example, for the PWL function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 2 - 3sgn(x), \qquad (3)$$

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[1] Actually, in the great majority of known examples, with g being polynomial, it is also a smooth function.

the set $\mathcal{M} = \{0\}$ determines the continuity (also the differentiability) sub-domains $\mathcal{D}_1 = (-\infty, 0)$, $\mathcal{D}_2 = (0, \infty)$ (see the graph shown in Fig. 1 a).

Definition 3 Let $x : [0, T] \to \mathbb{R}$ and $q > 0$. The *Caputo fractional derivative with starting point 0*, introduced by M. Caputo in 1967 in [15], is defined as

$$D_*^q x(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-\tau)^{n-q-1} x^{(n)}(\tau) d\tau. \qquad (4)$$

The fractional order, $q \in (n-1, n)$ with $n \in \mathbb{Z}^+$, is the nearest integer bigger than q ($n = \lceil q \rceil$), and Γ is Euler's Gamma function, a generalization of the factorial function $n!$, i.e. $\Gamma(n) = (n-1)!$, $n \in \mathbb{N}$, defined as

$$\Gamma(z) = \int_0^t t^{z-1} e^{-t} dt, \quad z \in \mathbb{C}, \quad Re(z) > 0.$$

Compared to other fractional-order differential operators, D_*^q is more restrictive since it requires the the existence of n-th (*first, for $q < 1$*) derivative of f (see e.g. [16–18]). However, the Caputo derivative with starting point 0 has the great advantage that it uses the fractional order initial conditions. Therefore, the use of Caputo's derivative in the IVP (1) is fully justified because in practical (physical) problems, we need physically interpretable initial conditions (see e.g. [18,19]). Thus, in (1), the initial condition(s) can be used as the integer-order differential equation counterpart, which, for any common case of $q \in (0, 1)$, will reduce to $x(0) = x_0$.

For the FDE associated with (3)

$$D_*^q x = 2 - 3sign(x), \quad x(0) = x_0, \qquad (5)$$

there are no classical (continuously differentiable) solutions starting from any point x_0. Thus, for $x = x_0 = 0$, there is no solution ($D_*^q(0) = 0 \neq 2 = 2 - 3sign(0)$). For $x_0 > 0$, there exists a solution but only on the interval $[0, T']$ with $T' = (\Gamma(1+q)x_0)^{1/q}$. This solution has the form $x(t) = x_0 - t^q/\Gamma(1+q)$, but it cannot be extended to any interval larger than $[0, T')$. For $x_0 < 0$, there also exists some $T'' = (\Gamma(1+q)x_0/5)^{1/q}$, such that the solution $x(t) = x_0 + 5t^q/\Gamma(1+q)$ exists but only on $[0, T'']$. Even if these solutions tend to the line $x = 0$, they cannot extend along this line (see Fig. 2 a, where $q = 0.6$ and $q = 0.8$).

One way to overcome this difficulty is similar to the one for integer DEs with discontinuous right-hand sides (Fillipo equations), namely to transform the discontinuous right-hand side into a convex set-valued function with closed values. In this way, the problem is restarted as a differential inclusion of fractional order.

Definition 4 A *set-valued* (*multi-valued*) function $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a function which associates to any element $x \in \mathbb{R}^n$, a subset of $\mathbb{R}^n$, $F(x)$ (the image of x).

There are several ways to define $F(x)$. The (convex) definition was introduced by Filippov in [20] (see also [21–23])

$$F(x) = \bigcap_{\varepsilon > 0} \bigcap_{\mu(\mathcal{M}) = 0} \overline{conv}(f(z \in \mathbb{R}^n : |z - x| \leq \varepsilon \backslash \mathcal{M})). \tag{6}$$

Here, $F(x)$ is the closure of the convex hull of $f(x)$, with ε being the radius of the ball centered at x. At any continuity point of f, $F(x)$ consists of one single point, which coincides with the value of f at this point (i.e. we get back $f(x)$ as the right-hand side: $F(x) = \{f(x)\}$), while at the points belonging to $\mathcal{M}$, $F(x)$ is given by (6).

If s_i are *sign* functions, the underling set-valued form, denoted by $Sign : \mathbb{R} \rightrightarrows \mathbb{R}$, is defined as follows

$$Sign(x) = \begin{cases} \{-1\}, & x < 0, \\ [-1, 1], & x = 0, \\ \{+1\}, & x > 0. \end{cases} \tag{7}$$

Thus, $sign(0)$ is taken now as the whole interval $[-1, 1]$ connecting the points -1 and $+1$.

In Fig. 3 a and Fig. 3 b, there are plotted *sign* and *Sign*, respectively, and in Fig. 2 b, the graph of the set-valued function corresponding to Example (3).

In this way, a discontinuous FDE of the form

$$D_*^q x = f(x), \quad x(0) = x_0, \quad t \in [0, T], \tag{8}$$

can be restarted as a set-valued problem (fractional-order differential inclusion (FDI))

$$D_*^q x \in F(x), \quad x(0) = x_0, \quad \text{for a.a.} \quad t \in [0, T]. \tag{9}$$

Differential inclusions of integer order have been intensively studied in the literature since the 1930s, one of the first works being attributed to the Polish mathematician Zaremba [24] (see also [20–23]), while the study of fractional-order differential inclusions was initiated by El-Sayed and Ibrahim in [25]. Some of the (few) works on differential inclusions of fractional order are the papers [26–28].

Following the way proposed by Filippov for DE of integer order with discontinuous right-hand side, a (generalized or Filippov) solution to (8) can be defined as follows ([20] p.85; see also [29]).

Definition 5 A *generalized solution* to (8) is an absolutely continuous function $x : [0, T] \to \mathbb{R}$, satisfying (9) for a.a. $t \in [0, T]$.

Thus, by applying Filippov's regularization, the problem (3) is converted to the following set-valued IVP of fractional order

$$D_*^q(x) - 2 \in -3Sign(x) = \begin{cases} 5, & x < 0, \\ [-1, 5], & x = 0, \\ -1, & x > 0, \end{cases} \tag{10}$$

for a.a. $t \in [0, T]$. Now, for $x_0 = 0$, one can choose for D_*^q any value in $[-1, 5]$, for example 0 (Fig. 1 b), and the equation at this point reads $D_*^q x = 0$. This shows that $x(t) = 0$, for $x_0 = 0$, is a solution which verifies now the equation. In this way, using (10), the FDE (5) will have a generalized solution. Therefore, for whatever initial condition x_0, the solution will reach the line $x = 0$ (dotted line in Fig. 2 b) and also continue forward.

Since the theory of numerical methods for FDI is only at the begining ([30] is one of the very few works on this subject), we propose a different approach to integrate numerically the underlying set-valued IVP: to approximate the set-valued problem with a single-valued continuous one of fractional order, for which there are known numerical methods, e.g. the predictor-corrector Adams-Bashforth-Moulton (ABM) method [31].

Definition 6 Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued function. The single-valued function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a *selection* (*approximation*) of F if $f(x) \in F(x)$ for every $x \in \mathbb{R}^n$ (see e.g. [32], [33]).

Definition 7 The map F is *upper semi-continuous* (USC) on $\mathbb{R}^n$ if for each $x_0 \in \mathbb{R}^n$, the set $F(x_0)$ is a nonempty and closed subset of $\mathbb{R}^n$, and if for each open set N of $\mathbb{R}^n$ containing $F(x_0)$, there exists an open neighborhood M of x_0 such that $F(M) \subset N$.

Theorem 1 (Cellina's Theorem [22,23]**).** *Let* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *be USC. If the values of* F *are nonempty and convex, then for every* $\varepsilon > 0$, *there exists a locally Lipschitz selection* $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$Graph(f_\varepsilon) \subset Graph(F) + \varepsilon B,$$

Remark 1 Due to the symmetric interpretation of a set-valued map as a graph (see e.g. [22]), we shall say that a set-valued map satisfies a property if and only if its graph satisfies it. For instance, a set-valued map is said to be convex if and only if its graph is a convex set.

For the set-valued function $F(x) = 2 - 3Sign(x)$, a sketch of a selection is plotted in Fig. 1 c.

3 Lyapunov exponents

In this section, we show that the discontinuous IVP (1) can be continuously approximated. Thus, by applying the Filippov regularization to (1) leads to the following FDI

$$D_*^q x \in F(x) := g(x) + A(x)S(x), \qquad (11)$$

with

$$S(x) = (S_1(x_1), S_2(x_2), ..., S_n(x_n))^T, \qquad (12)$$

where $S_i : \mathbb{R} \to \mathbb{R}$ are the set-valued variants of s_i, $i = 1, 2, ..., n$ ($Sgn(x_i)$ for the usual case of $sgn(x_i)$).

The existence of solutions for FDIs is analyzed in, e.g., [25]. However, because of the lack of numerical methods to find the solutions to fractional-order differential inclusions, necessary to determine LEs, we need to convert the set-valued IVP to a single-valued one of fractional order.

It is easy to check that F, defined in (11), verifies the requirements of Theorem 1 (see [34]). Therefore, we can enounce the following theorem:

Theorem 2 [13] *The PWC IVP of fractional order (1) can be transformed to the following continuous IVP of fractional order,*

$$D_*^q x = f_\varepsilon(x) := g(x) + A(x)s_\varepsilon(x), \qquad (13)$$

where $s_\varepsilon(x) = (s_{1\varepsilon}(x_1), s_{2\varepsilon}(x_2), ..., s_{n\varepsilon}(x_n))^T$ is a continuous approximation of $S(x) = (S_1(x_1), S_2(x_2), ..., S_n(x_n))^T$.

The steps of the proof are sketched in Fig. 4. As can be seen, the constructive proof allows to choose the approximation simply by replacing s with a continuous function s_ε (grey line in Fig. 4).

The existence of (Lipschitz) continuous approximations for the set-valued function F defined in (11) is studied in [35].

The approximation of f defined in (1) can be done as closely as desired, and can be made locally (in small neighborhoods of discontinuity points of s), or globally (in small neighborhoods of the graph of S; see Fig. 1 c) [13].
Generally, a set-valued function admits (infinitely) many local or global approximations.

In what follows, we use global approximations, which are easy to implement numerically.

For the sake of simplicity, for all component $s_{i\varepsilon}(x_i)$, $i = 1, 2, ..., n$, ε can have the same value.

Since most of practical examples of PWC systems are modeled via the *sign* function, we use for its global approximation the so-called *sigmoid function*, $sign_\varepsilon$, because this class of functions provide the required flexibility and to which the abruptness of the discontinuity can be easily modified[2]

$$sign_{\varepsilon(\delta)}(x) = \frac{2}{\pi} arctan \frac{x}{\delta} \approx Sign(x). \qquad (14)$$

In (14), δ is a positive parameter which controls the slope of the function in the ε-neighborhood of the discontinuity $x = 0$ (In Fig. 5 a, the graph is plotted for a large value of δ, $\delta = 1E - 1$, for a clear image).

For global approximations, ε is determined implicitly, depending proportionally on the δ size, $\varepsilon = \varepsilon(\delta)$. For example, in order to obtain a neighborhood ε of order $1E - 6$, we need to choose for δ the value $1E - 4$ (in [13], a detailed numerical analysis for the case of the sigmoid function, $sign_\varepsilon(x) = 2/(1 + e^{-x/\delta}) - 1$, is presented).

The function in example (3) becomes

$$f(x) \approx f_{\varepsilon(\delta)}(x) = 2 - 3sign_{\varepsilon(\delta)}(x) = 2 - \frac{6}{\pi} arctan \frac{x}{\delta}. \ (15)$$

As for the case of integer-order systems, in order to determine the Lyapunov spectrum, we show next that it is possible to find a differential (variational) equation of fractional order.

Let us consider again the IVP (1). The following theorem holds:

Theorem 3 *System (1) has the following variational equations which define the LEs*

$$\begin{aligned} D_*^q \Phi(t) &= D_x f_\varepsilon(x)\Phi(t), \\ \Phi(0) &= I_n. \end{aligned} \qquad (16)$$

Proof By Theorem 2, f is approximated by f_ε. Under the adopted assumptions, there exists a flow [36] $\phi : I \times \mathbb{R}^n \to \mathbb{R}^n$ of the approximated system (13), $x = \phi(t, x)$, which satisfies $D_*^q \phi(t, x_0) = f_\varepsilon(\phi(t, x_0))$, $\phi(0, x_0) = x_0$, for all $t \in I$. Next, Theorem 2 in [6] ensures the existence of the variational equation (16), where $D_x f_\varepsilon(x)$ is the matrix of the partial derivatives of f_ε evaluated along the solution $x = \phi(t, x)$[3], $\Phi(t) = D_x \phi(t, x)$ is the Jacobian of the flow ϕ evaluated at $\phi(t, x)$ and I_n is the identity matrix.

The parallel solving of systems (13) and (16), which is a linear matrix-valued and time-varying system with coefficients depending on the evolution of the original system (11), allows us to determine LEs.

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[2] This class of sigmoid functions includes many other examples such as the hyperbolic tangent, the error function, the logistic function, algebraic functions like $\frac{x}{\sqrt{\delta + x^2}}$, $\frac{2}{1 + e^{-\frac{x}{\delta}}} - 1$ [13] and so on.

[3] Because, while the problem is solved in parallel with (13), the initial conditions change, so the usual notation x_0 is replaced with x.

Remark 2 This result can be extended to Cauchy problems involving Riemann-Liouville derivative [6]. However, due to the benefit of using initial conditions in integer order, we restrict the study to Caputo's derivative.

Oseledec's Multiplicative Ergodic Theorem [1], which ensures the existence of LEs from the stability matrix of the system, still applies to fractional-oredr systems, and it proves that the entire spectrum of LEs can be determined and the following limit exists and is finite

$$\lambda_k = \lim_{t \to \infty} \frac{1}{t} ln|\Lambda_k(t)|, \quad k = 1, 2, ..., n. \tag{17}$$

In (17), $\Lambda_k(t)$ are the eigenvalues of $\Phi(t)$ and, following the common order convention, the ordered LEs: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ measure the rate of growth of infinitesimal k-dimensional volumes, $k = 1, ..., n$.

Because the analytic evaluation is rarely available, usually the LEs are computationally estimated.

Therefore, by replacing the new formulas in the known algorithms for LEs spectrum or for the largest LE, the numerical method for ODEs of integer order (usually the 4th RK method) with some methods for FDEs (ABM method in this paper), one can determine the LEs spectrum, the largest LE or their bounds (see e.g. [6] or [7] for LEs spectrum and largest LE).

4 Applications

Even it is difficult and not recommended in practical systems to determine the LEs, their utility as means of predicting any observable of physical significance being still contestable (see e.g. [37], Chp. 6), the determination of LEs exponents remains a widely used tool to characterize chaotic behavior. We now consider three examples, which are in concordance with the theoretical result of this paper.

The LE spectrum is determined with the Matlab code *lyapunov.m* [38], which adapts Wolf's algorithm [39] for continuous systems of integer order to calculate the spectrum of LEs from systems where, as in the considered examples, the equations are known as well the largest Lyapunov exponent from an experimental time series. For this purpose, in Wolf's algorithm, the ODEs integrator for continuous flows (usually the classical RungeKutta method) is replaced with some numerical schemes for FDEs.[4]

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[4] In this paper, the Matlab code *fde12.m* [40] (an implementation of the predictor-corrector PECE method of Adams-Bashforth-Moulton type presented in [31], which has been suitably modified for the general case with the incommensurate orders) has been utilized.

As is known, the numerical results concerning the LEs are subject to several uncertainties related to the utilized numerical method and especially to the initial conditions. Therefore, we calculated the LEs several times with slightly altered initial conditions and then considered the mean and standard deviation of the results (the graphs showing the LEs dynamics present one of this evolution).

The integration step size is $h = 0.001$ and the dynamics of the LE spectrum were determined for $t \in [0, 400]$, while the chaotic attractors were drawn for $t \in [0, 1000]$.

Considering the usual case of the PWC *sign* function, in order to integrate the variational equation (16), we need to calculate the derivative of the sigmoid $sign_\varepsilon$

$$sign'_{\varepsilon(\delta)}(x) = \frac{d}{dx} sign_{\varepsilon(\delta)}(x) = \frac{2}{\pi} \frac{\delta}{\delta^2 + x^2}. \tag{18}$$

At $x = 0$, due to the vertical segment, the set-valued function $Sign$ has slope $+\infty$, while any approximation $f_{\varepsilon(\delta)}$ of $Sign$ has a finite slope given by the derivative at $x = 0$, as

$$m = sign'_{\varepsilon(\delta)}(0) = \frac{2}{\pi\delta}.$$

The graphs of $sign_{\varepsilon(\delta)}$ and $sign'_{\varepsilon(\delta)}$, chosen for clarity $\delta = 1E - 1$, are plotted in Fig. 4.

For δ, we consider $\delta = 5E - 4$. For this value, ε is of order of $1E - 6$, and $m = 4000/\pi$. Lower values for δ imply higher values for the derivative of $sign_\varepsilon$ at $x = 0$. For example, for $\delta = 1E - 5$, $m = 2E5/pi$ which, considering the roundoff errors and also the errors of ABM method [31,41], can finally lead to a loss of precision. On the other hand, higher values imply modifications only on the third decimals of the LEs values.

– The Sprott system (2), for $q = (0.99, 0.98, 0.97)$ and the chosen parameters $a = 1$ and $b = 0.5$, behaves chaotically (Fig. 6) and has the approximated form

$$\begin{aligned} D_*^{0.99} x_1 &= x_2, \\ D_*^{0.98} x_2 &= x_3, \\ D_*^{0.97} x_3 &= -x_1 - x_2 - 0.5x_3 + sign_{\varepsilon(\delta)}(x_1), \end{aligned} \tag{19}$$

with $sign_{\varepsilon(\delta)}(x_1)$ given by (14), and the Jacobian

$$D_x f_{\varepsilon(\delta)}(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 + sign'_{\varepsilon(\delta)}(x_1) & -1 & -0.5 \end{pmatrix},$$

where $sign'_{\varepsilon(\delta)}$ is given by (18).

The LE spectrum, obtained after running the code on $I = [0, 400]$ (see the dynamics of LEs in Fig.7), is $\{0.11, -0.01, -0.73\}$.

– Let us next consider the fractional-order variant of the piecewise linear (PWL) Chen's system [42]

$$D_*^{q_1} x_1 = a (x_2 - x_1),$$
$$D_*^{q_2} x_2 = sign(x_1)(c - a - x_3) + cdx_2, \quad (20)$$
$$D_*^{q_3} x_3 = x_1 sign(x_2) - bx_3,$$

with coefficients $a = 1.18$, $b = 0.16$, $c = 1.2$, $d = 0.1$, where

$$g(x) = \begin{pmatrix} a(x_2 - x_1) \\ cdx_2 \\ -bx_3 \end{pmatrix}, \quad A(x) = \begin{pmatrix} 0 & 0 & 0 \\ c - a - x_3 & 0 & 0 \\ 0 & x_1 & 0 \end{pmatrix}.$$

For $q = (0.99, 0.9, 0.999)$, and for the considered coefficients values, the system behaves chaotically (Fig. 8), and the approximated form is

$$D_*^{0.99} x_1 = 1.18 (x_2 - x_1),$$
$$D_*^{0.9} x_2 = sign_{\varepsilon(\delta)}(x_1)(0.82 - x_3) + 0.12x_2, \quad (21)$$
$$D_*^{0.999} x_3 = x_1 sign_{\varepsilon(\delta)}(x_2) - 0.16x_3,$$

which has the Jacobian

$$D_x f_{\varepsilon(\delta)}(x) =$$

$$\begin{pmatrix} -1.18 & 1.18 & 0 \\ 0.82 sign'_{\varepsilon(\delta)}(x_1) & 0.12 & -sign_{\varepsilon(\delta)}(x_1) \\ sign_{\varepsilon(\delta)}(x_2) & x_1 sign'_{\varepsilon(\delta)}(x_2) & -0.16 \end{pmatrix}.$$

The LEs are: $\{0.71, 0.28, -2.19\}$ (see Fig. 9). Because there are two positive exponents, the PWC Chen's system of fractional order is hyperchaotic.

– The last example is the fractional-order variant of the PWC Shimizu–Morioka's system [43,44] with $q = (0.99, 0.97, 0.98)$

$$D_*^{0.99} x_1 = x_2,$$
$$D_*^{0.97} x_2 = (1 - x_3)sign(x_1) - ax_2, \quad (22)$$
$$D_*^{0.98} x_3 = x_1^2 - bx_3,$$

which with $a = 0.75$ and $b = 0.45$ behaves chaotically (Fig. 10). Here, $g(x) = (x_2, -0.75x_2, x_1^2 - 0.45x_3)^T$ and

$$A(x) = \begin{pmatrix} 0 & 0 & 0 \\ 1 - x_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The approximated system is

$$D_*^{0.99} x_1 = x_2,$$
$$D_*^{0.97} x_2 = (1 - x_3)sign_{\varepsilon(\delta)}(x_1) - 0.75x_2, \quad (23)$$
$$D_*^{0.98} x_3 = x_1^2 - 0.45x_3,$$

and the Jacobian has the following form

$$D_x f_{\varepsilon(\delta)}(x) =$$

$$\begin{pmatrix} 0 & 1 & 0 \\ (1 - x_3)sign'_{\varepsilon(\delta)}(x_1) & -0.75 & -sign_{\varepsilon(\delta)}(x_1) \\ 2x_1 & 0 & -0.45 \end{pmatrix}.$$

The LEs are: $\{0.23, 0.13, -1.82\}$ (see Fig. 11). Similar to Chen's system, Shimizu–Morioka's system is hyperchaotic due to the presence of two positive LEs.

All the above examples present chaotic behaviors. Let us finally consider the PWC Shimizu–Morioka's system (22) of order $q = (0.8, 0.7, 0.6)$. The equilibrium points are $X^*(\pm\sqrt{b}, 0, 1)$ which are asymptotically stable [45]. Therefore, as expected, all the LEs are negatives are: $(-0.001, -0.002, -0.040)$ (Fig. 12). This result is in concordance with the known tendency of chaos disappearance in the case of chaotic systems of fractional order, for order decreasing.

5 Conclusion

In this paper, we have shown that PWC systems of fractional order have well-defined LEs. To prove that the PWC systems can be continuously approximated, Cellina's Theorem and some results from the theory of differential equations with discontinuous right-hand sides have been utilized.

The approximation of the discontinuous elements on the right-hand sides of the IVPs, which generally are *sign* functions, has been realized with the sigmoid function $2/\pi atan(x/\delta)$, with δ being a parameter which determines the slope of approximation in small neighborhoods of the discontinuity point $x = 0$.

The variational equations which define the LEs help to find numerically the LEs. For this purpose, we used Matlab implementations of the known Wolf algorithm in which the numerical method for ODEs has been replaced with the predictor-corrector ABM method for fractional-order differential equations.

A future task related to this subject would be the numerical analysis of the computational errors given by any algorithm for numerical determination of LEs and also given by the numerical methods for FDEs. As is well known, the errors in these cases are quite large and, therefore, an optimal integration step size correlated to the maximal integration interval would be of practical importance.

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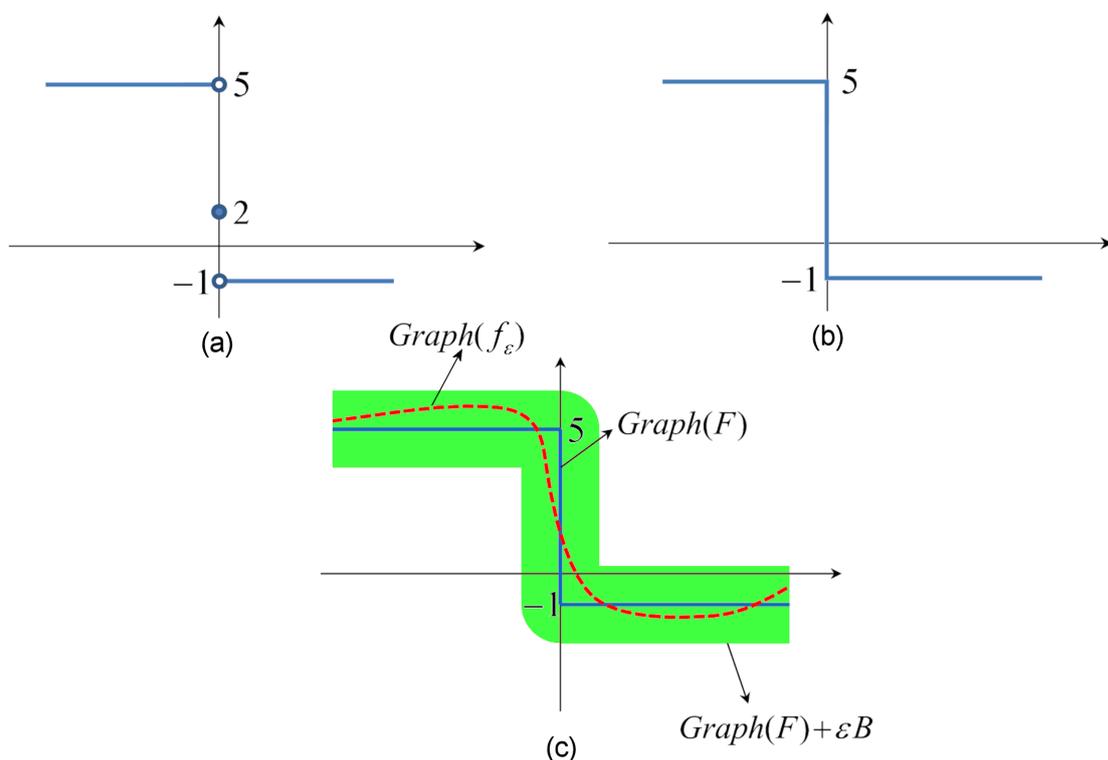
Fig. 1 Graph of the function $f(x) = 2 - 3sign(x)$. a) Discontinuity at $x = 0$. b) Graph of the set-valued version of f. c) Sketch of a continuous selection of the set-valued function F within an ε-neighborhood of F.
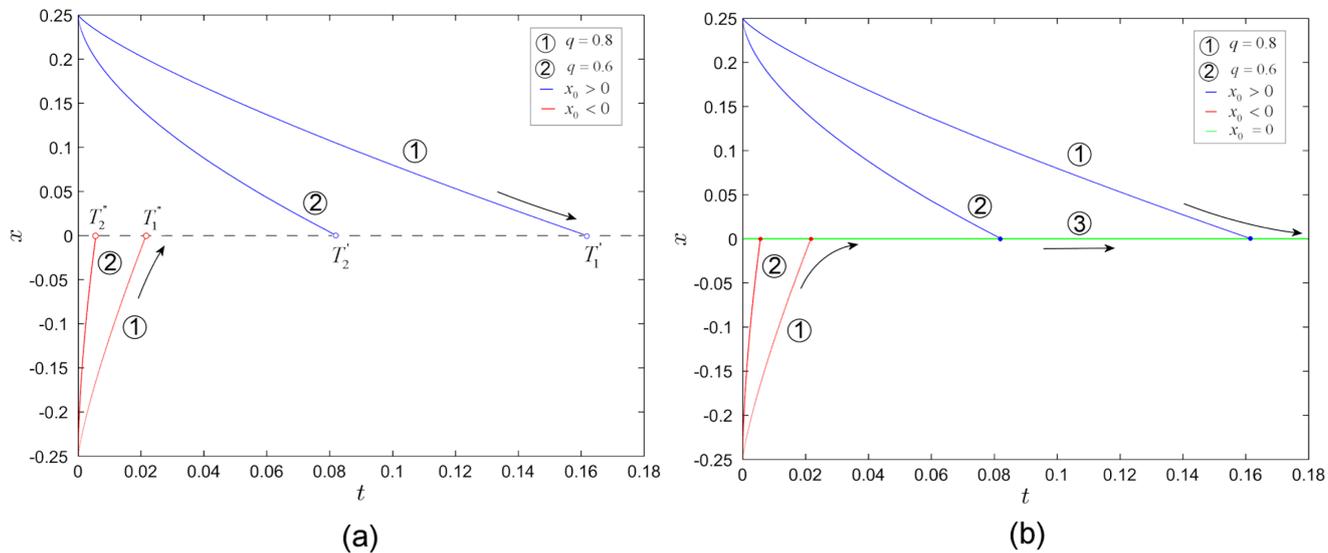
Fig. 2 Solutions of the equation $D_*^q x = 2 - 3sign(x)$ for $q = 0.8$ and $q = 0.6$. a) Depending on the sign of x_0, the solutions tend to the line $x = 0$ but they cannot reach this line; there is no a classical solution. b) Considering the differential inclusion $D_*^q x - 2 \in -3Sign(x)$, there exists a generalized solution which continues through the line $x = 0$ for whatever x_0.
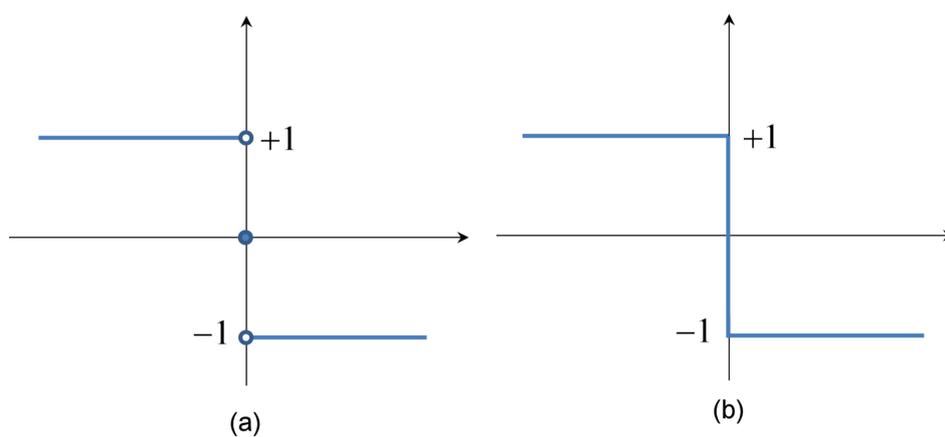
Fig. 3 a) Graph of $sign(x)$. b) Graph of $Sign(x)$.
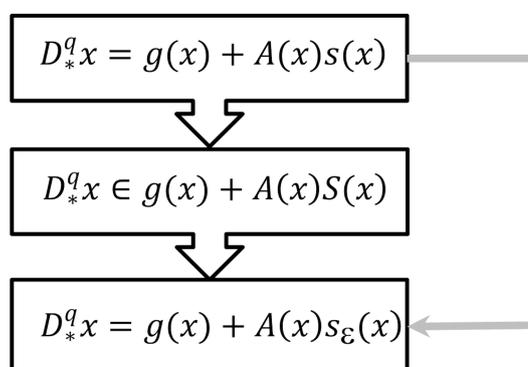
Fig. 4 The sketch of the proof steps of Theorem 2.
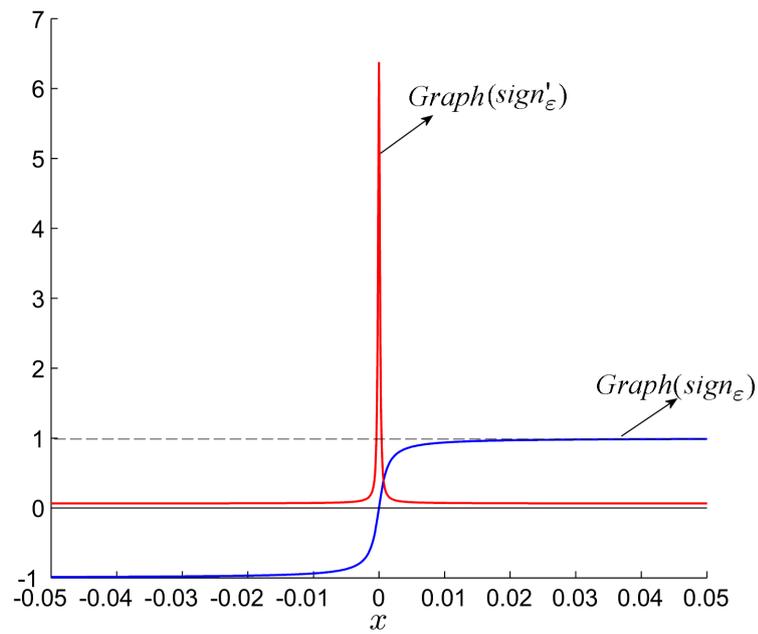
Fig. 5 Graphs of $sign_{\varepsilon(1e-1)}$ (blue plot) and $\frac{d}{dx}sign_{\varepsilon(1e-1)}$ (red plot).
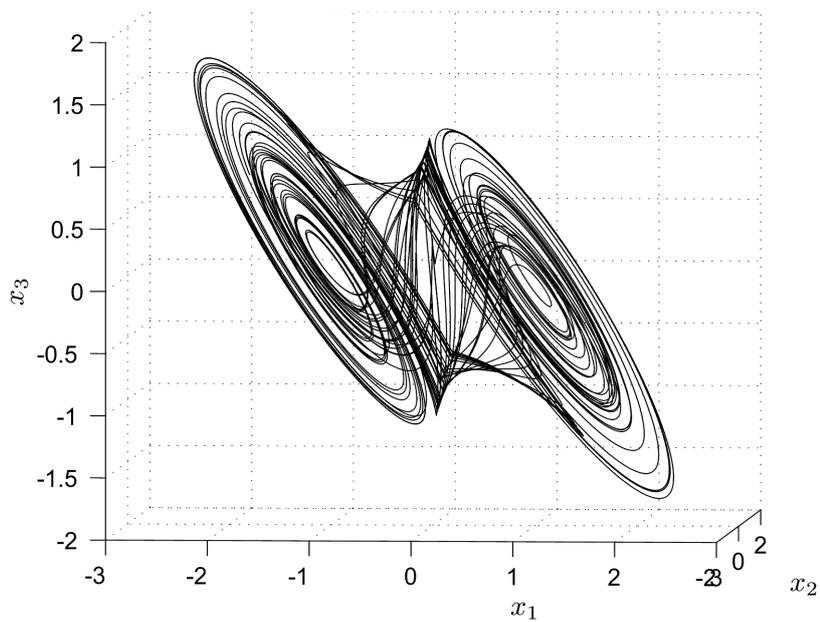
Fig. 6 Sprott's chaotic attractor.
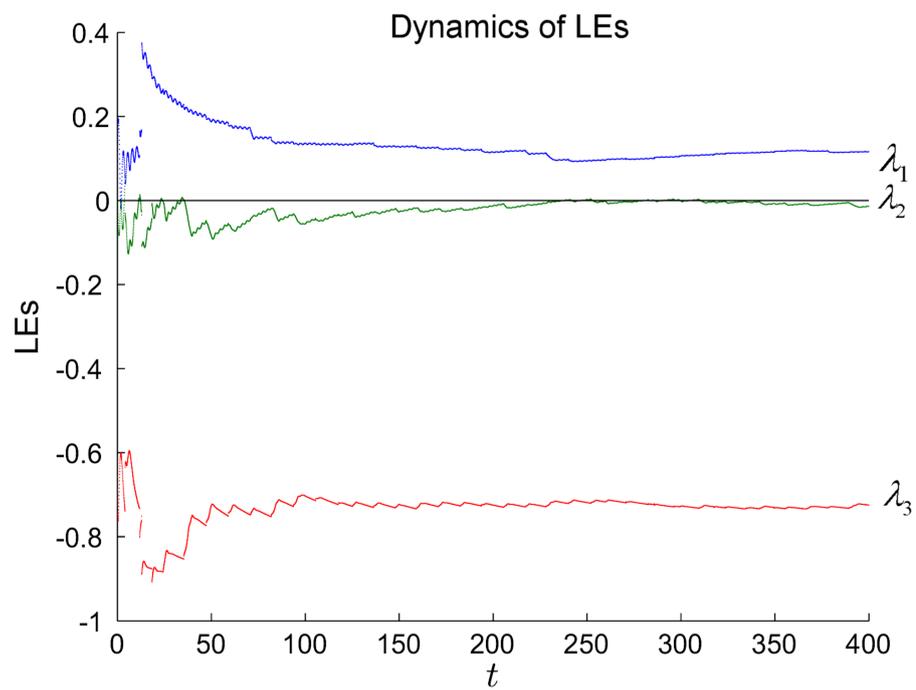
Fig. 7 Dynamics of the LEs exponents for the Sprott system for $q = (0.99, 0.98, 0.97)$.
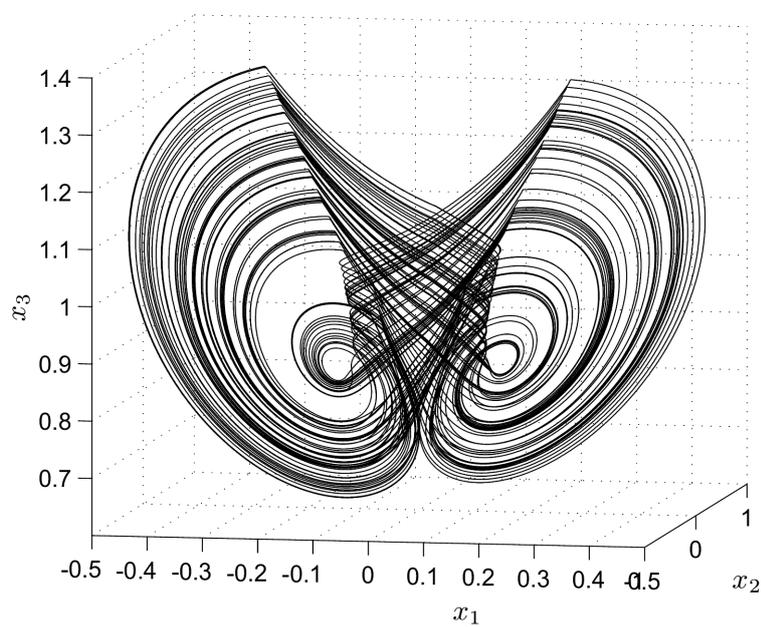
Fig. 8 Chen's chaotic attractor.
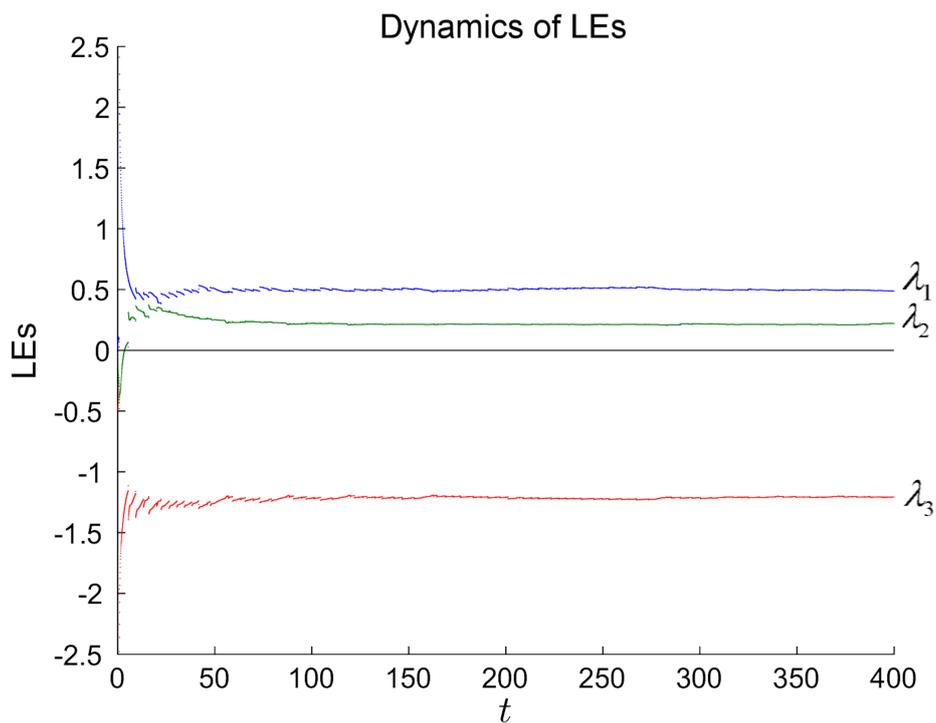
Fig. 9 Dynamics of the LEs exponents for the Chen system for $q = (0.99, 0.9, 0.999)$.
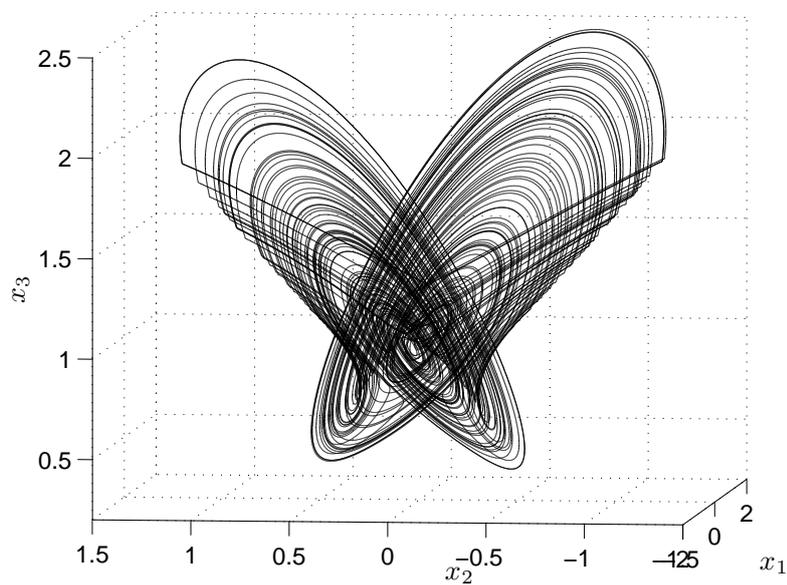
Fig. 10 Shimizu–Morioka's chaotic attractor.
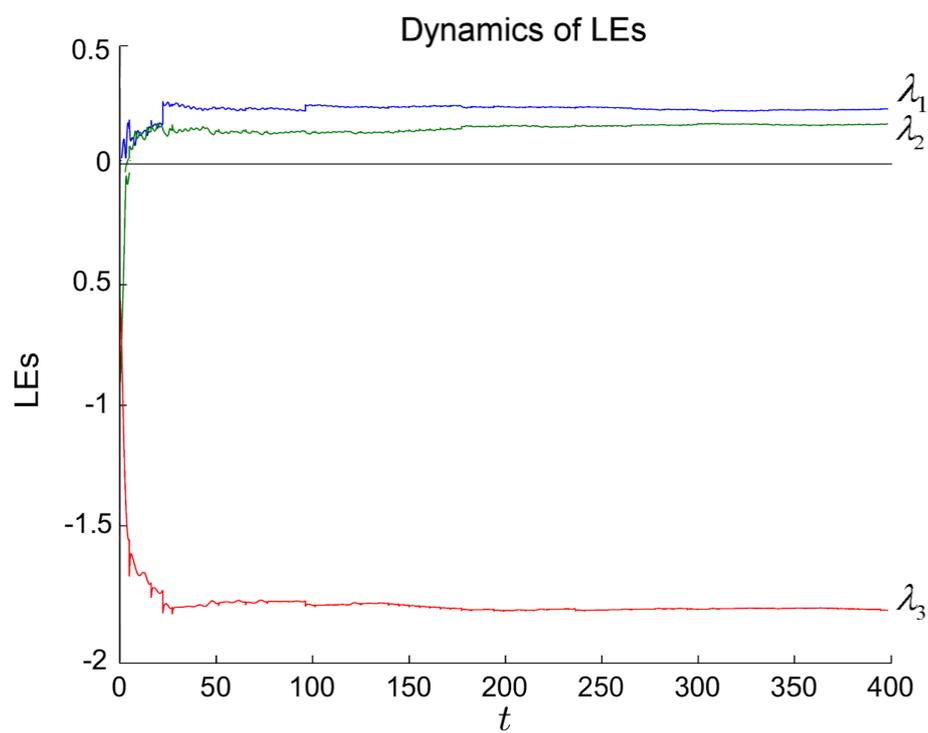
Fig. 11 Dynamics of the LEs exponents for the Shimizu–Morioka system for $q = (0.99, 0.97, 0.98)$.
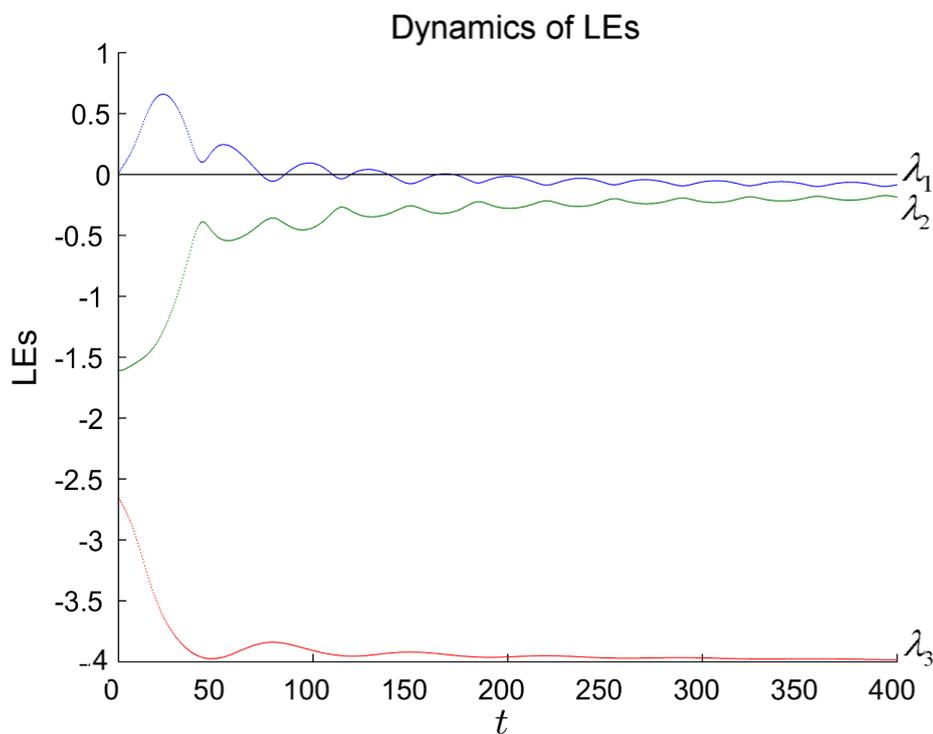
Fig. 12 Dynamics of the LEs exponents for the Shimizu–Morioka system for $q = (0.8, 0.7, 0.6)$.