CHAOS **18**, 033111 (2008)

Random parameter-switching synthesis of a class of hyperbolic attractors

Marius-F. Danca
Department of Applied Sciences, Avram Iancu University, Cluj-Napoca, Romania

(Received 12 March 2008; accepted 8 July 2008; published online 4 August 2008; publisher error corrected 14 August 2008)

The parameter perturbation methods (the most known being the OGY method) apply small wisely chosen swift kicks to the system once per cycle, to maintain it near the desired unstable periodic orbit. Thus, one can consider that a new attractor is finally generated. Another class of methods which allow the attractors born, imply small perturbations of the state variable [see, e.g., J. Güémez and M. A. Matías, Phys. Lett. A **181**, 29 (1993)]. Whatever technique is utilized, generating any targeted attractor starting from a set of two or more of any kind of attractors (stable or not) of a considered dissipative continuous-time system cannot be achieved with these techniques. This kind of attractor synthesis [introduced in M.-F. Danca, W. K. S. Tang, and G. Chen, Appl. Math. Comput. **201**, 650 (2008) and proved analytically in Y. Mao, W. K. S. Tang, and M.-F. Danca, Appl. Math. Comput. (submitted)] which starts from a set of given attractors, allows us, via periodic parameter-switching, to generate any of the set of all possible attractors of a class of continuous-time dissipative dynamical systems, depending linearly on the control parameter. In this paper we extend this technique proving empirically that even random manners for switching can be utilized for this purpose. These parameter-switches schemes are very easy to implement and require only the mathematical model of the underlying dynamical system, a convergent numerical method to integrate the system, and the bifurcation diagram to choose specific attractors. Relatively large parameter switches are admitted. As a main result, these switching algorithms (deterministic or random) offer a new perspective on the set of all attractors of a class of dissipative continuous-time dynamical systems. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2965524]

In Ref. 1, the attractors of a considered dissipative continuous-time system were synthesized using deterministic manners for parameter switches. In this paper we extend these results and prove numerically that the parameter-switching techniques work even if random switching manners are utilized. The synthesized attractor is identical to one of the existing attractors obtained by integration of the mathematical model for a precise parameter value. Moreover, these techniques (deterministic or random) reveal a vector spacelike structure of the hyperbolic attractors. Numerical simulations illustrate that a wide range of attractors can be obtained by this scheme. Since any of the existing attractors can be synthesized with these techniques, chaos control and anticontrol can be viewed as attractor synthesis via parameter-switching techniques.

I. INTRODUCTION

Let us consider a class of continuous-time autonomous dissipative dynamical systems depending linearly on a single real parameter, modeled by the following general initial value problem (I.V.P.):

$$S: \dot{x} = f_p(x), \quad x(0) = x_0, \tag{1}$$

where $p \in \mathbb{R}$ and $f_p: \mathbb{R}^n \to \mathbb{R}^n$ has the expression

$$f_p(x) = g(x) + pAx, \tag{2}$$

with $g: \mathbb{R}^n \to \mathbb{R}^n$ a continuous-time nonlinear function, A a real constant $n \times n$ matrix, $x_0 \in \mathbb{R}^n$, and $t \in I = [0, \infty)$.

It is supposed that the existence and uniqueness of solutions on the maximal existence interval $I = [0, \infty)$ are assumed, and that there exists only hyperbolic equilibria.

The synthesis of hyperbolic attractors of dynamical systems modeled by I.V.P. (1) by periodic parameter-switching is presented in Ref. 1. This technique, empirically proved, continues a sequence of works dealing with the parameter switches. Thus, switching schemes was introduced in Refs. 2–4.

Empirically proved by various experiments, a desired attractor of a considered system modeled by Eq. (1), can be duly numerically synthesized by a proposed switching deterministic time-varying scheme; each step-size of the numerical method to I.V.P. (1), p is switched in a deterministic manner between a set of chosen values of p.

The obtained attractor is one of the existing attractors belonging to the set of all possible attractors of the system modeled by Eq. (1). Moreover, the most interesting is the fact that the synthesized attractor can be obtained directly by numerical integration of Eq. (1) for p given by a precisely linear combination of the considered switching values of the parameter.

In this paper we extend this subject and prove numerically that even random manners of the parameter switches lead to the same result: any hyperbolic attractor depending on p can be synthesized by these kind of switches, either deterministic or stochastic.

The paper is organized as follows: In the next section, the deterministic parameter-switching scheme is described. In Sec. III the random synthesis of the attractors is presented. Finally, in Sec. IV, some concluding remarks are given and some issues for future works are discussed.

II. DETERMINISTIC SYNTHESIS

Notation 1: *Let $\mathcal{A}$ the set of all global attractors depending on parameter p* (the attractors not depending on p are not considered), *including attractive stable fixed points, limit cycles, and chaotic attractors. Let also $\mathcal{P}\subset\mathbb{R}$ be the set of the corresponding admissible values of p. Denote by $\mathcal{P}_N=\{p_1,p_2,\ldots,p_N\}\subset\mathcal{P}$ a finite ordered subset of $\mathcal{P}$ containing N different values of p, which determines the set of attractors $\mathcal{A}_N=\{A_{p_1},A_{p_2},\ldots,A_{p_N}\}\subset\mathcal{A}$.*

As it is known, for some fixed initial condition, a convergent numerical method simulates one of the local attractors belonging to the global attractor (see the Appendix). More precisely, the *ω-limit set* is obtained (see the Appendix). Therefore, in this paper by *attractor* one understands its *ω-limit set*, actually its approximation, which as usual,[5] is considered after neglecting a sufficiently long period of transients.

Because of the dissipativity, $\mathcal{A}$ is nonempty. It then follows naturally that a bijection between $\mathcal{P}$ and $\mathcal{A}$, can be defined. Thus, giving any $p\in\mathcal{P}$, there exists a unique attractor, and vice versa.

A major aspect in this paper is to compare the numerically synthesized attractors. The geometric structure of attractors can be very complicated. Therefore, it is extremely difficult, if not impossible, to determine the position of a chaotic attractor in the phase space. Also that appears to be true even for an equilibrium point or a periodic trajectory in general.

Recognizing these difficulties in comparing attractors, the following simple and practical criterion is introduced.

Criterion 2: *Two attractors are considered to be* (almost) identical *if*

(i) *their geometrical forms in the phase space coincide*;
(ii) *the sense of the motion in time is preserved.*

Criterion 2 represents a suitable modification of the known concept of *topological equivalence* (see, e.g., Ref. 6) being useful for practical use rather than for theoretical rigor.

This above geometrical identity concept, aided by phase representations, histograms, and Poincaré sections, serves well for the attractors computer graphic inspection. However the situation becomes complicated for chaotic attractors (see, e.g., Ref. 7). In this case, the "almost" identity would be justified by a geometric coincidence of their *branched manifolds* (see Ref. 8) near the preserved sense of motion in time on the trajectories.

Remark 3: *(i) The term* almost *in criterion 2 refers simply to the case of chaotic attractors where the similarity between two chaotic attractors may arise only asymptotically for $t\to\infty$.*

(ii) Using criterion 2, the invariance under the changes of control-parameter values of branched manifolds is avoided, and thus, the objectivity between $\mathcal{P}$ and $\mathcal{A}$ follows logically. Also, the use of some inherent tools of topological characterization (considering, for example, the shape of an attractor, it is possible to have two attractors possessing the same shape and however being different in the sense of criterion 2) *or dimensions related to the comparison of attractors (see, e.g., Refs. 6, 7, and 9–11) can be avoided.*

Given a convergent numerical method for I.V.P. (1) and a convergent numerical method, for a fixed step-size h, the following conjecture can be introduced, the analytical proof being presented in Ref. 12.

Conjecture 4: *For any finite set $\mathcal{A}_N$ of attractors, corresponding to P_N there exists an attractor A^* generated by numerical integration of Eq. (1) with switching parameter p in P_N upon certain rules. Moreover, $A^*\in A$, and A^* is (almost) identical to an attractor $A_p\in A$ corresponding to a specific value p given by*

$$p=\frac{\sum_{k=1}^{N}p_{\varphi(k)}m_k}{\sum_{k=1}^{N}m_k}. \tag{3}$$

In order to see what m_k and $p_{\varphi(k)}$ do represent let us consider a partition of I, $I=\cup_{i\in\mathbb{N}^*}[t_{i-1},t_i)$, with $t_0=0$, such that $t_i=jh$, for $i,j\in\mathbb{N}$, where h is the integration step of the considered numerical method. Then, the switching synthesis rule in conjecture 4 can be codified as the following: $(m_1+m_2+\cdots+m_N)h$-periodic sequence:

$$[m_1p_{\varphi(1)},\quad m_2p_{\varphi(2)},\ldots,m_Np_{\varphi(N)}], \tag{4}$$

where the weights m_i are some positive integers and φ permutes the subset $\{1,2,\ldots,N\}$.

It is known that under a variety of Lipschitz conditions some numerical methods for ODEs defines a dynamical system.[13] Thus, under the assumptions on the existence and uniqueness on the I.V.P. (1), we are entitled to compare the dynamical system defined by the numerical approximation to I.V.P. (1) with the underlying dynamical system itself.

Scheme (4) represents the deterministic time-periodic way to synthesize the hyperbolic attractors of the dynamical system modeled by I.V.P. (1) and has the following significance: the considered numerical method will integrate Eq. (1) with $p=p_{\varphi(1)}$ for the first m_1 steps, and then with $p=p_{\varphi(2)}$ in the next m_2 steps, and so on, until the last Nth subinterval. Then the cycle is repeated on the next N subintervals so that a periodic parameter-switching scheme is obtained.

For example, the sequence $[2p_1,3p_3,5p_2]$ indicates that, for the first 2 integration steps, $p=p_1$, and then for the next 3 integration steps, $p=p_3$, and for the last 5 steps, $p=p_2$. After that, the cycle is repeated again, i.e., $[2p_1,3p_3,5p_2]$ should be understood as being the following periodical sequence:

$$2p_1,3p_3,5p_2,2p_1,3p_3,5p_2,\ldots. \tag{5}$$

Example 5: *For example, the scheme (5) applied to the Lorenz attractor for $p_1=125$, $p_2=140$ and $p_3=175$ corresponding to chaotic movements [Figs. 1(a)–1(d)] with the weights $m_1=2$, $m_2=3$ and $m_3=3$ and $h=0.0005$ using the standard Runge–Kutta method, generates the attractor A^*.*

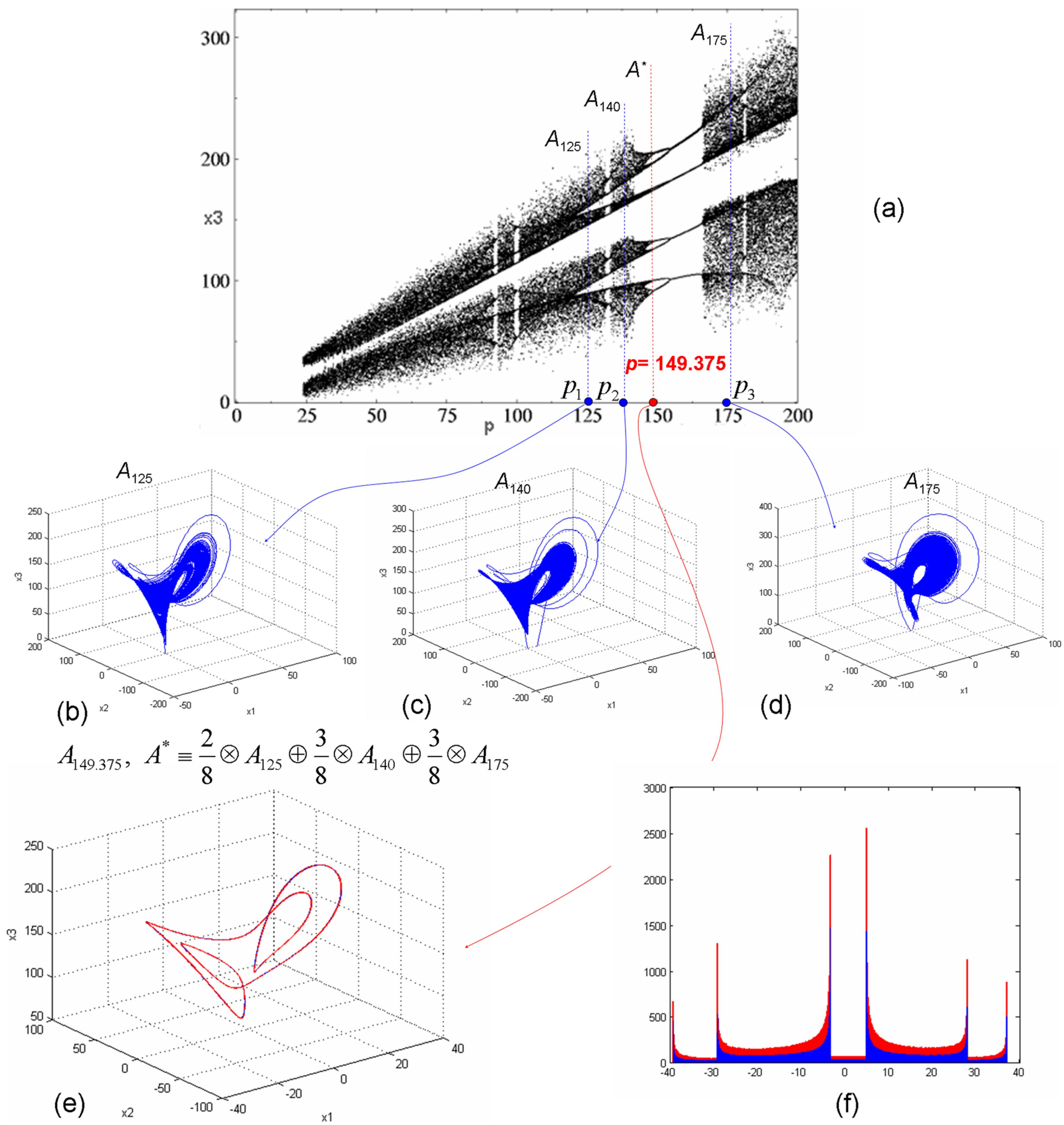

FIG. 1. (Color online) Periodic stable attractor (limit cycle) of the Lorenz system. (a) Bifurcation diagram. (b)–(d) Phase plots of three chaotic attractors corresponding to $p_1=125$, $p_2=140$, and $p_3=175$. (e) The synthesized attractors A^* and A_p, with $p=149.375$, superimposed. (f) Histograms of A^* and A_p superimposed.

Moreover, A^* is identical to the attractor A_p with p given by the formula (3), i.e., $p=(m_1p_1+m_2p_3+m_3p_2)/(m_1+m_2+m_3)=149.375$. The phase portrait and histogram, superimposed [*Figs. 1(e) and 1(f)*] *show this identity.*

If we denote $\alpha_k=m_k/\Sigma_{k=1}^N m_k$, α_k verifies the affine combination, $\Sigma\alpha_k=1$ and based on the bijection between $\mathcal{P}$ and $\mathcal{A}$ and the following formula (4), we could endow $\mathcal{A}$ with two binary abstract operations $\oplus$ and $\otimes$ such that conjecture 4 could be reformulated as follows:

Conjecture 6: *$A \in \mathcal{A}$, if and only if there exist N positive integers $m_k, k=1,\ldots,N$, such that*

$$A = \alpha_1 \otimes A_{p_1} \oplus \alpha_2 \otimes A_{p_2} \oplus \cdots \oplus \alpha_N \otimes A_{p_N},$$

$$A_{p_k} \in \mathcal{A}, \quad \alpha_k = m_k \Big/ \sum_{k=1}^{N} m_k. \tag{6}$$

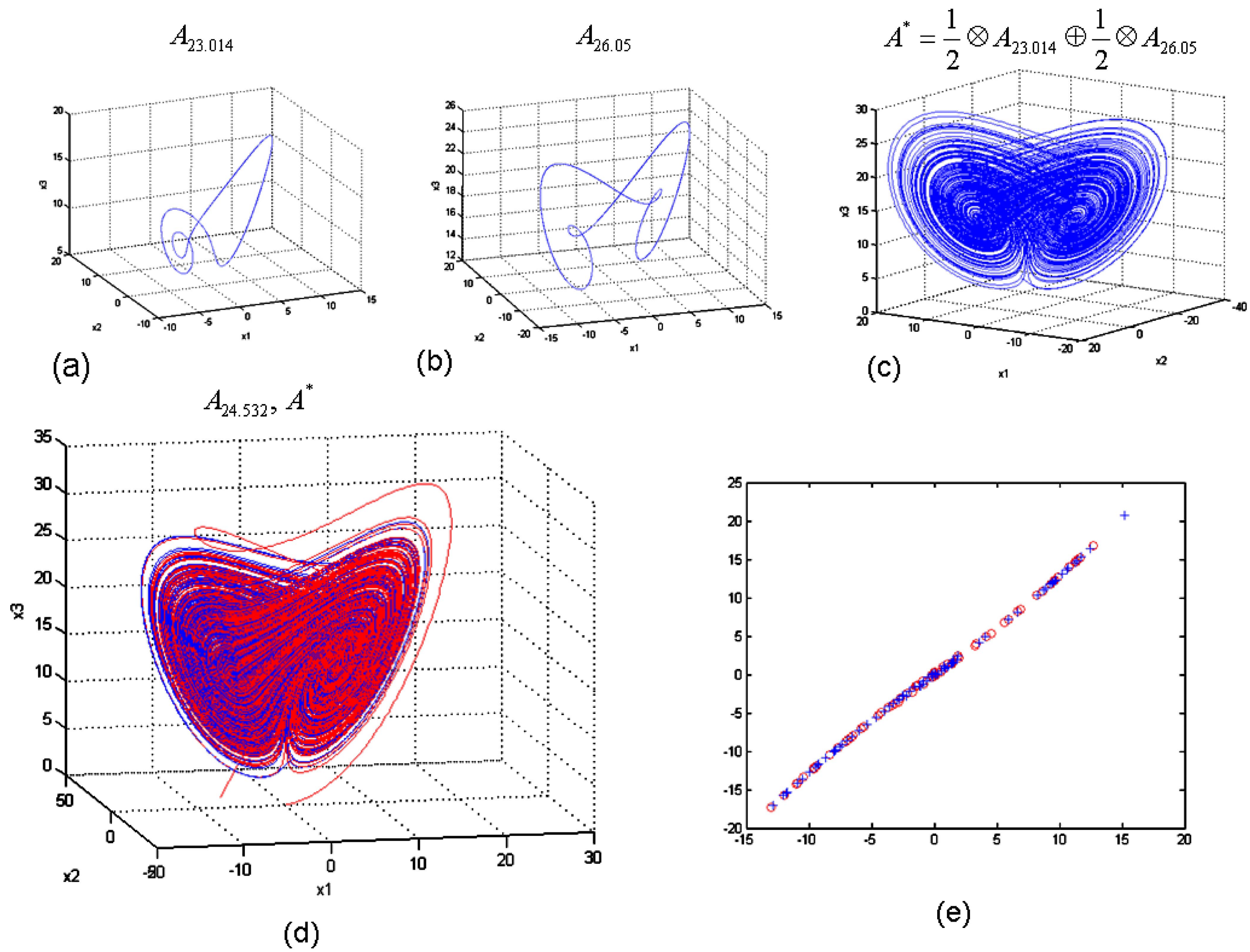

FIG. 2. (Color online) Synthesis of a chaotic Chen attractor. (a), (b) Two periodic attractors. (c) The synthesized attractor. (d) A_p and A^* superimposed. (e) Poincaré sections of A_p and A^* superimposed.

In other words, if one considers a set $\mathcal{A}_N$ of N attractors there exists a set of N positive integers m such that the right-hand side of the affine relation (6) generates, via Eq. (4), an attractor A^* which is almost identical to an attractor $A \in \mathcal{A}$ corresponding to a specific parameter $p \in \mathcal{P}$ and reversely for any attractor A there exist a set of N positive integers m and a set $\mathcal{A}_N$ of N attractors such that A can be decomposed as in Eq. (6).

Remark 7: *(i) If p is a rational nonterminating (repeating) decimal number, or has the decimal number greater than the computer internal representation, due to computational numerical errors, some relative small differences can appear in between the two attractors, A^* and A_p.*

(ii) Let an ordered set $\mathcal{P}_N = \{p_{\min}, \ldots, p_{\max}\}$. The resultant averaged p, given by Eq. (3), is located, obviously, inside the interval $(p_{\min}, p_{\max})$, i.e., $p_{\min} < p < p_{\max}$. Thus, if $\mathcal{P}_N$ is chosen within a chaotic/periodic band in the bifurcation diagram, the resultant attractor will also be chaotic/periodic. But, if $\mathcal{P}_N$ covers disjoint bands, the resultant attractor could be of any type. For example, if $N=2$, A^* will be situated, in the parameter space, between the attractors A_{p_1} and A_{p_2}. In this manner, the control and anticontrol of chaos can be achieved (see Ref. 1).

(iii) Generally, for fixed x_0 and h, Eq. (4) is not "commutative," i.e., $[m_1p_1, m_2p_2]$ and $[m_2p_2, m_1p_1]$ generally give different attractors.

The deterministic scheme (4) can be described algorithmically as described in Algorithm 1.

Algorithm 1

repeat
for $k=1$ *to* m_1 *do*
integrate IVP with $p=p_1$
for $k=1$ *to* m_2 *do*
integrate IVP with $p=p_2$
$\vdots$
for $k=1$ *to* m_N *do*
integrate IVP with $p=p_N$
$t=t+h$
until $t \geq T$

Example 8: *The scheme (4) can be utilized as an anticontrol technique in the following way: suppose we want to obtain a chaotic Chen attractor (see the Appendix) A^* starting from two stable periodic attractors A_{p_1} and A_{p_2} [Figs. 2(a) and 2(b)] corresponding, e.g., to $p_1=23.014$ and*

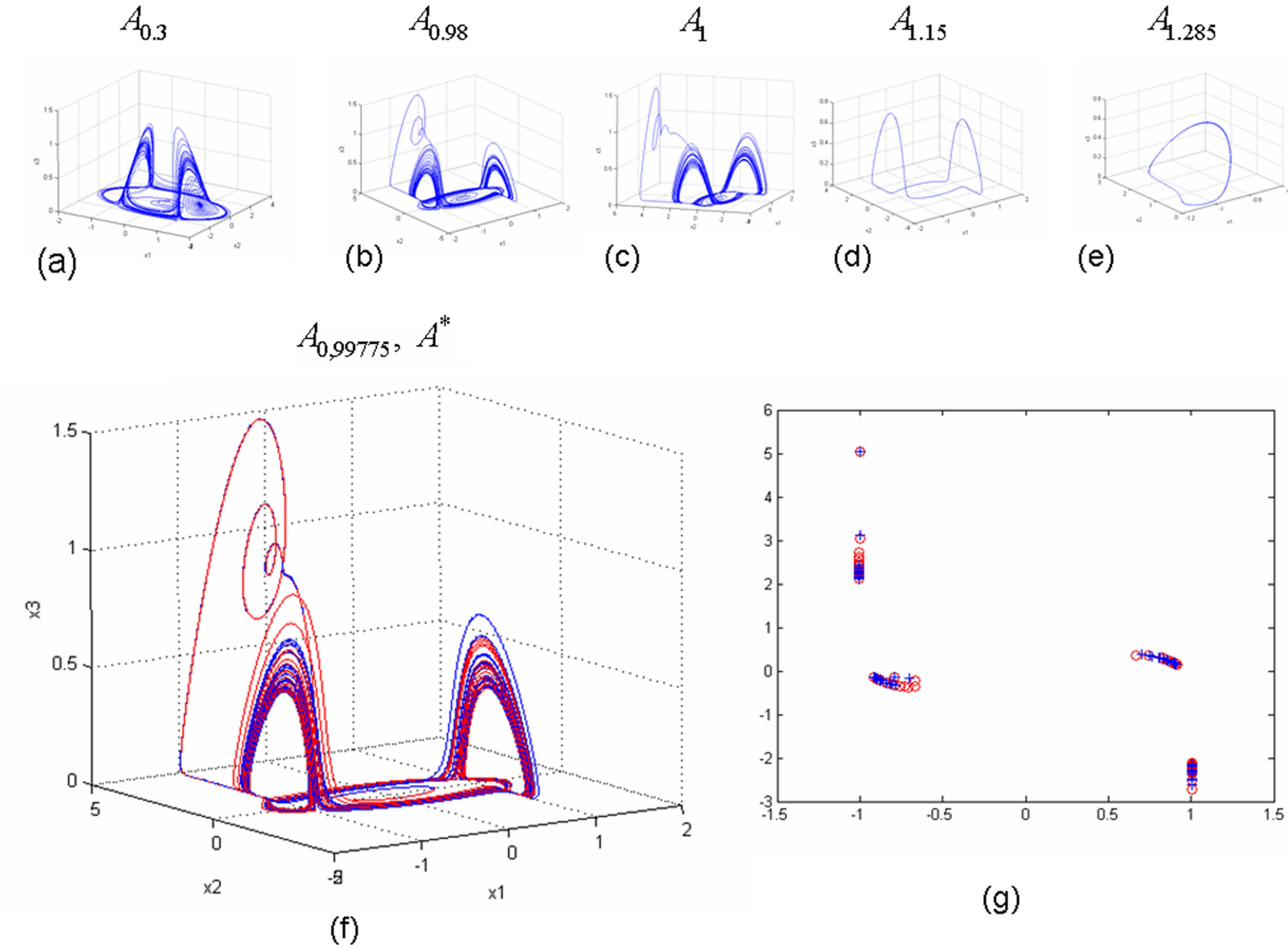

FIG. 3. (Color online) Deterministic synthesis of a chaotic Rabinovich–Fabrikant system. (a)–(e) Five attractors. (f) The synthesized attractors A_p and A^* superimposed. (g) Superimposed Poincaré sections of A_p and A^*.

$p_2=26.05$. *Because between* p_1 *and* p_2 *there is a relative large chaotic window in the bifurcation scenario, with* $m_1=m_2=1$ *the resulting* p *given by Eq. (3),* $p=(p_1+p_2)/(m_1+m_2)=24.532$ *indicates a chaotic attractor. Thus the synthesized attractor* A^* *[Fig. 2(c)] obtained with the sequence* $[1p_1,1p_2]$ *is (almost) identical to* A_p*. In Fig. 2(d), both* A^* *and* $A_{24.532}$ *are plotted superimposed, while in Fig. 2(e) the Poincaré section underlines the identity.*

Example 9: *A chaotic attractor for the Rabinovich–Fabrikant system (see the Appendix) and which represents a real challenge to integration numerical methods, can be obtained even with a complicate scheme* $[2p_1,3p_5,4p_4,5p_2,6p_3]$ *with* $p_1=0.3$, $p_2=0.98$, $p_3=1$, $p_4=1.15$, *and* $p_5=1.285$ *[Figs. 3(a)–3(e)]. The synthesized attractor* A^* *is almost (see Remark 3 i) identical to* $A_{0.99775}$*, where* $0.99775=(2p_1+3p_5+4p_4+5p_2+6p_3)/20$ *[Figs. 3(f) and 3(g)].*

Summarizing, one can imagine the following relationships:

$$\mathcal{D}(p_1,p_2,\ldots,p_N)\rightarrow A^*\xrightarrow{(3)}A_p,$$

where $\mathcal{D}$ represent some deterministic and periodic rule for changing p_i when integrating I.V.P. (1).

III. RANDOM SYNTHESIS

While in the above examples the synthesis scheme was applied in a deterministic manner, random choices too in Eq. (4) confirm his rightness. Thus, one of the first random verified variants is that when p_i, $i=1,\ldots,N$ are chosen in a random order in Eq. (4) (see Algorithm 2).

Algorithm 2
$m_{p_i}=0$, $i=1,\ldots N$
while $t<T$ *do*
label=rand(N)
case label of
1: *GOTO label* 1
2: *GOTO label* 2
…
N: GOTO label N
label 1: *for* $k=1$ *to* m_1 *do*
integrate IVP with $p=p_1$
inc(m_{p_1})
label 2: *for* $k=1$ *to* m_2 *do*
integrate IVP with $p=p_2$
inc(m_{p_2})
…

label N: *for* $k=1$ *to* m_N *do*
integrate IVP with $p=p_N$
$inc(m_{p_3})$
$t=t+h$

where $\mathrm{rand}(N)$ generates a random number between 1 and N and m_{p_i} represent the counter for integration of IVP for each p_i, $i=1,\ldots,N$. In this case for p we have the following relation:

$$p=\frac{m_{p_1}p_1+m_{p_2}p_2+\cdots+m_{p_N}p_N}{m_{p_1}+m_{p_2}+\cdots+m_{p_N}}. \tag{7}$$

Example 10: *For the Rabinovich–Fabrikant system (see the Appendix) if one chose* $N=3$, $p_1=1.285$, $p_2=1.01$, $p_3=1.195$, $m_1=2$, $m_2=3$, $m_3=4$, *the resulted attractor* A^* *is identical to* A_p *with* p *given by Eq. (7)* $p=(m_{p_1}p_1+m_{p_2}p_2+m_{p_3}p_3)/(m_{p_1}+m_{p_2}+m_{p_3})=1.145\,855\,361\,285\,536\,12$. *In Fig. 4(a),* $A_{1.14\ldots}$ *and* A^* *are plotted superimposed, while in Fig. 4(b) the histograms superimposed are presented. With* $h=0.001$, $T=400$ *from Algorithm 2 one obtains* $m_{p_1}=63\,972$, $m_{p_2}=12\,940$ *and* $m_{p_3}=60\,944$.

Another simple possibility to implement randomness is to attribute to p the values p_i for a random number of steps size h (see Algorithm 3).

Algorithm 3
repeat
for $k=1$ *to* $\mathrm{rand}(m_1)$ *do*
integrate IVP with $p=p_1$
$inc(m_{p_1})$
for $k=1$ *to* $\mathrm{rand}(m_2)$ *do*
integrate IVP with $p=p_2$
$inc(m_{p_2})$
$\vdots$
for $k=1$ *to* $\mathrm{rand}(m_N)$ *do*
integrate IVP with $p=p_N$
$inc(m_{p_N})$
$t=t+h$
until $t\geq T$

Example 11: *For example, for the Rossler system (see the Appendix) with* $p_1=18$, $p_2=25$, $p_3=31$, *and* $m_1=2$, $m_2=4$ *and* $m_3=3$, *the synthesized attractor* A^* *is almost identical to* A_p *(Fig. 5) with* $p=26.365\,452\,691$ *obtained with Eq. (7). For this system, special attention should be paid because of his sensitivity on the computed results as pointed out in Ref. 14.*

Example 12: *A stronger stochastic way could be obtained if both the order of* p_i *and the number of times when* $p=p_i$ *are changed randomly (i.e., combining Algorithm 1*

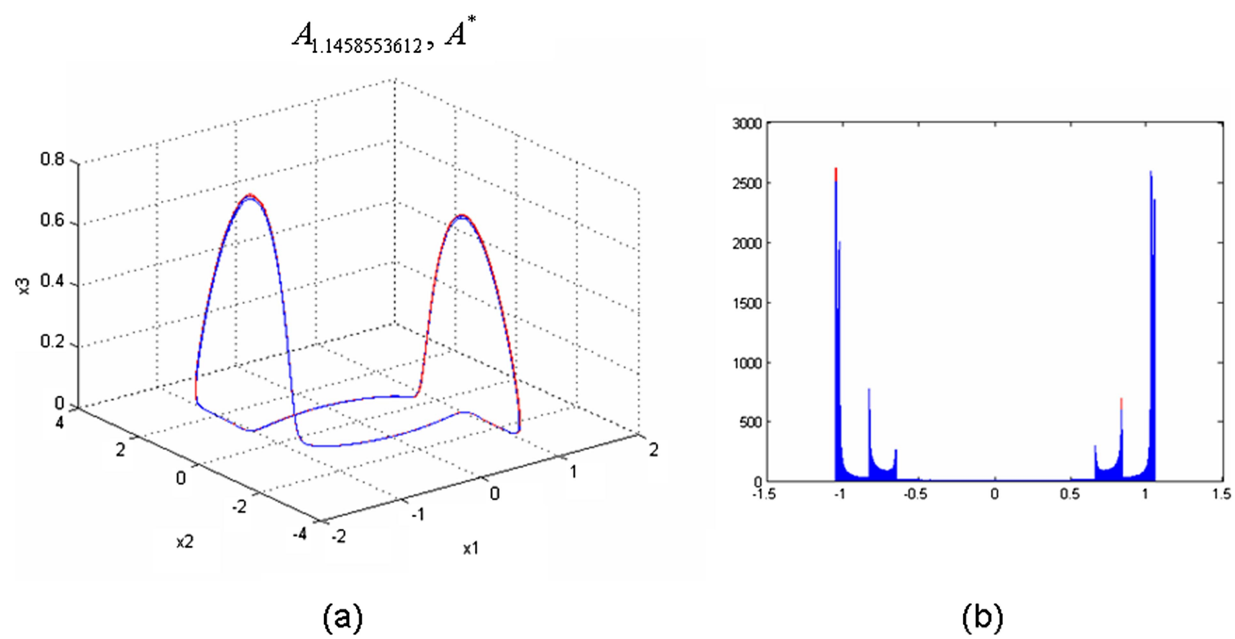

FIG. 4. (Color online) Random synthesis of a limit cycle of the Rabinovich–Fabrikant system. (a) A^* and A_p superimposed. (b) Histograms of A^* and A_p superimposed.

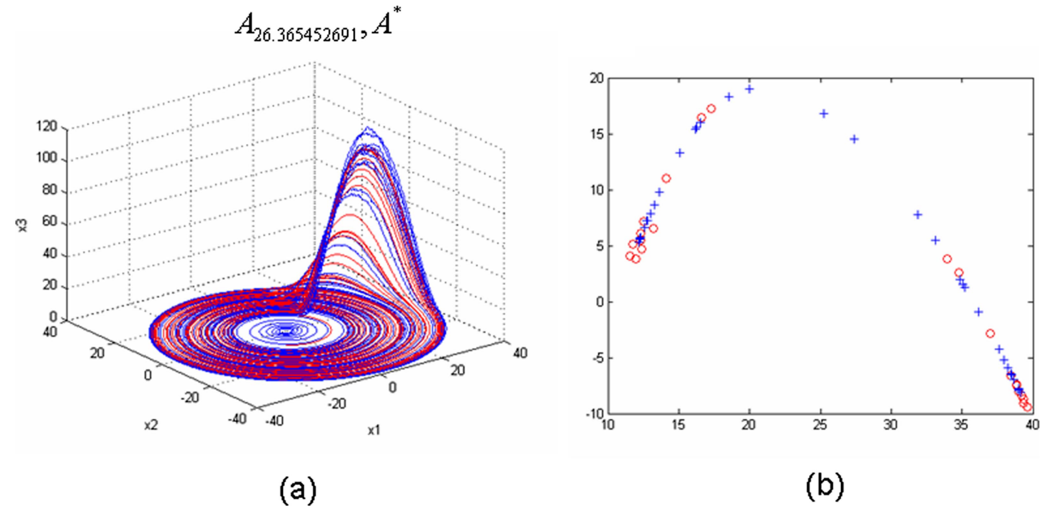

FIG. 5. (Color online) Random synthesized attractor of the Rössler system. (a) A_p and A^* superimposed. (b) Superimposed Poincaré sections of A_p and A^*.

and 2). In this case for the Chen system with $p_1=23$,$p_2=24$, and $p_3=32$ and $m_1=7$, $m_2=3$ and $m_3=4$, A^* is (almost) identical to $A_{24.067\,659\,198}$ (Fig. 6).

For the random manners of dealing with p_i in Eq. (4) one can imagine the following relationships:

$$\mathcal{R}(p_1,p_2,\ldots,p_N) \rightarrow A^* \overset{(7)}{\rightarrow} A_p, \tag{8}$$

where $\mathcal{R}$ represents some random rule to fix p_i.

Remark 13: *(i) The functions used in our simulations to generate random numbers are the known random function with uniform distribution existing in all compilers.*

(ii) In the cases of the random schemes, the limitations of the internal representations, may lead to relative small difference between A^* and A_p.

(iii) Let $\mathcal{R}_i$, $i=1,2,\ldots,M$ be M random different algorithms and $A_i^*=\mathcal{R}_i(p_1,p_2,\ldots,p_N)$, $i=1,2,\ldots,M$ [relation (8)]. Then $A_j^* \neq A_k^*$ for any $j\neq k$, $j,k\in\{1,\ldots,M\}$. This obviously follows from Eq. (7). Thus, for example, for Chen's attractor with $p_1=23$, $p_2=24$ and $p_3=32$ using the Algorithm 2 one obtains $p=24.121\,540\,068$ while Algorithm 3 gives p $=24.067\,659\,198$.

(iv) While deterministic switches techniques can explain and even achieve the control and anticontrol of chaos, the random algorithms are useless in control or anticontrol since we cannot precisely calculate in advance the synthesized p.

(v) Let $m=\max\{m_1,\ldots,m_N\}$. If m has a relatively large value, then A^* still remains in a relatively small neighborhood of A_p but its trajectory presents some corners.

IV. CONCLUSIONS AND DISCUSSION

In this paper we have proved numerically that any hyperbolic attractor of a dynamical system modeled by the I.V.P. (1) can be considered synthesized via the periodic (or random) switching scheme (4) of p.

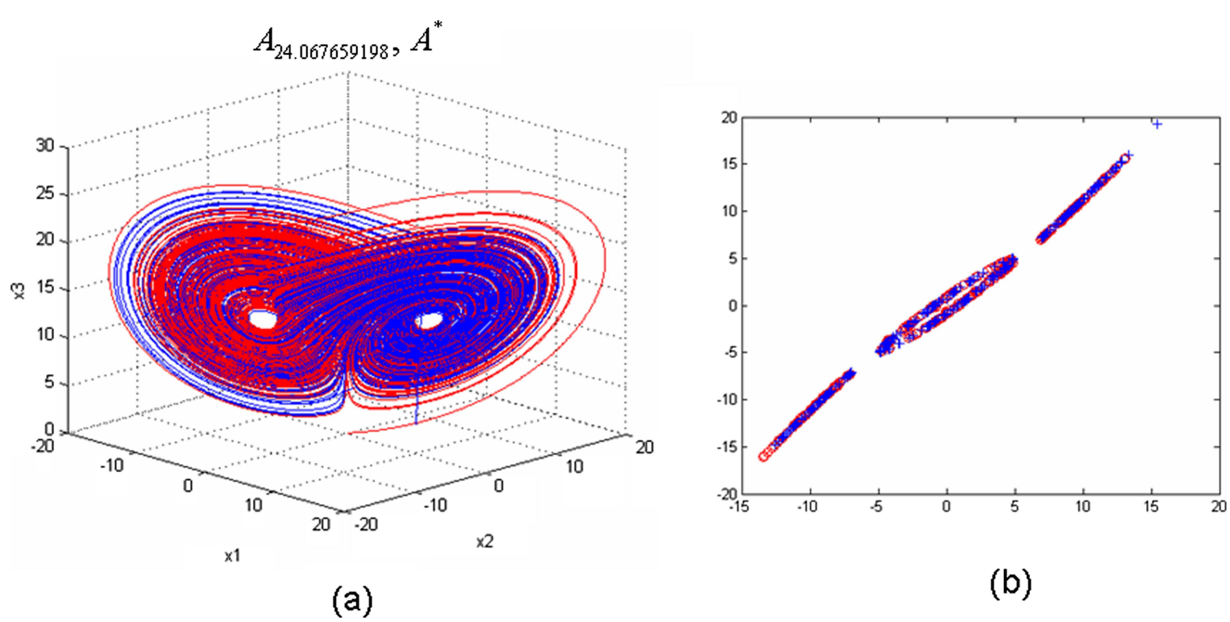

FIG. 6. (Color online) Random synthesized attractor of the Chen system. (a) A_p and A^* superimposed. (b) Superimposed Poincaré sections of A_p and A^*.

Supported by intensive simulations, the scheme (4) is viable for large limits of the weights m and for any admissible values of the parameter p.

The novel and main result presented in Ref. 1 and extended in the present paper is the fact that the set of all attractors of a dynamical system modeled by Eq. (1) can be regarded as a vector spacelike, where each vector (attractor) can be expressed as a combination of a finite set of different vectors (attractors). The analytical proof remains a task for a future work.

As an interesting fact, the deterministic parameter-switching (see Remark 13 iv) can be considered as a very simple and suggestive explanation of attractors born in the control and anticontrol of chaos, the only condition being the alternation of the order and chaotic windows in the bifurcation space, condition generally verified by the dynamical systems. Thus, in Example 5 between $p=125$ and $p=175$, there exist several chaotic and periodic windows. To realize the control/anticontrol for given attractors $A_{p_1}, A_{p_2}, \ldots, A_{p_N}$ in Eq. (3) one can fix a desired p and solve the equation in the two unknowns $m_1, \ldots, m_N$. For example, the Lorenz control (Fig. 1) was realized by choosing m_1 and m_2 and $p=150$ in a periodic window in the bifurcation diagram. Solving Eq. (3) one obtains $m_3=[80/25]=3$. Recalculating p one finally obtains $p=149.375$. Other methods to solve Eq. (3) can be used, as a function of the given attractors A_{p_i}, or m_i.

In the presented synthesis algorithm, the values for p which make the system unstable, can be chosen too because of relative short periods of time (m step-size with m relatively small number).

If one denotes by Γ a trajectory of a given dynamical system modeled by the I.V.P. (1), Γ can be symbolized, for a considered numerical method, step-size h, and the sets $\{p_1, p_2, \ldots, p_N\}$ and $\{m_1, m_2, \ldots, m_N\}$ as a periodic infinite sequence. For example, for the Chen's attractor $A_{26.25}$ the corresponding trajectory can be symbolized as follows:

$$\Gamma = p_1, p_2, p_1, p_2, \ldots .$$

Thus, apparently paradoxical, chaotic trajectories (and inherently their underlying chaotic attractors) can be represented as symbolized by an infinite periodic sequence of p and periodic trajectories can be symbolized as a stochastic sequence of p.

Because of the nonuniqueness of the representations of rational numbers (3) these representations are not unique; each p may have infinity fractions of Eq. (3) like representations.

One impediment is the fact that adaptive step-size numerical methods circumvent the use of Eq. (3).

APPENDIX: NOTIONS AND UTILIZED SYSTEM MODELS

First, we give the notions related to the global, local attractors and *ω-limit set*. Then the models of the dynamical systems considered in this are presented.

1. Attractors and ω-limit set

Definition 14: *A* global attractor *of S is a compact set composing of all bounded global trajectories.*

There is vast literature concerning the existence of global attractors especially in the field of PDEs (we mention, e.g., Refs. 15–19), but it is a useful notion for ODEs too.

From the definition, a global attractor contains all the dynamics evolving from all possible initial conditions. In other words, it contains all solutions, including stationary solutions, periodic solutions, as well as chaotic attractors, relevant to the asymptotic behaviors of the system.

Definition 15: *A* local attractor *is a compact set, invariant under f, which attracts its neighboring trajectories.*

A global attractor is hence considered as being composed of the set of all local attractors, where each local attractor only attracts trajectories from a subset of initial conditions, specified by its basin of attraction. Therefore, for a fixed parameter p, different local attractors may be obtained depending on the choice of the initial condition $\mathbf{x}_0$, in contrast to the uniqueness of the case of a single global attractor.

For example, if one considers the Lorenz system with $p=2.5$, there are three local attractors: the origin (saddle) and two symmetrical fixed points (sinks) $X_{1,2}(\pm 2, \mp 2, 1.5)$. In some cases, a unique local attractor may also be the global one. For example, when $p=28$, there exists only a single local attractor, which is a global attractor too (known as the Lorenz strange attractor).

When a global attractor is composed by several local attractors, the initial conditions are essential for the numerical approximations of one of these attractors.

Definition 16: *The* ω-limit set *of a trajectory through* $x \in \mathbb{R}^n$ *is given as* $\omega(x) = \cap_{s\geq 0} \cup_{t\geq s} \Phi(t,x)$.

2. Utilized dynamical systems

The dynamical equations of the four utilized systems are, as follows:

Chen's system:[20]

$$\dot{x}_1 = a(x_2 - x_1), \quad \dot{x}_2 = (p-a)x_1 - x_1x_3 + px_2,$$

$$\dot{x}_3 = x_1x_2 - bx_3,$$

with parameters $a=35$ and $b=3$, while p is chosen as the control parameter here.

Referring to Eq. (2), one has

$$g(x) = \begin{bmatrix} a(x_2 - x_1) \\ -x_1x_3 - x_2 \\ x_1x_2 - bx_3 \end{bmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Lorenz system:

$$\dot{x}_1 = a(x_2 - x_1), \quad \dot{x}_2 = x_1(p - x_3) - x_2,$$

$$\dot{x}_3 = x_1x_2 - cx_3,$$

with $a=10$ and $c=8/3$, and p again is the control parameter.

Rössler system:

$$\dot{x}_1 = -x_2 - x_3, \quad \dot{x}_2 = x_1 + ax_2,$$

$$\dot{x}_3 = b + x_3(x_1 - p),$$

with $a=b=0.1$, and p is the control parameter.

Rabinovich–Fabrikant:[21]

$$\dot{x}_1 = x_2(x_3 - 1 + x_1^2) + ax_1, \quad \dot{x}_2 = x_1(3x_3 + 1 - x_1^2) + ax_2,$$

$$\dot{x}_3 = -2x_3(p + x_1x_2),$$

where a is set to 0.1.

[1] M.-F. Danca, W. K. S. Tang, and G. Chen, Appl. Math. Comput. **201**, 650 (2008).
[2] J. Almeida, D. Peralta-Salas, and M. Romera, Physica D **200**, 124 (2005).
[3] M.-F. Danca and M. Romera, Dyn. Contin. Discrete Impulsive Syst.: Ser. B **14**, 155 (2008).
[4] M. Romera, M. Small, and M.-F. Danca, Appl. Math. Comput. **192**, 283 (2007).
[5] C. Foias and M. S. Jolly, Nonlinearity **8**, 295 (1995).
[6] J. K. Hale, *Topics in Dynamic Bifurcation Theory* (American Mathematical Society, New York, 1981), CBMS 47.
[7] B. Fiedler and C. Rocha, Trans. Am. Math. Soc. **352**, 257 (1999).
[8] D. A. Salamon, IAS/Park City Mathematics Series, Vol. 7, pp. 145–229 (1999).
[9] J. Christy, Trans. Am. Math. Soc. **336**, 759 (1993).
[10] L. Kapitanski and I. Rodnianski, Commun. Math. Phys. **53**, 218 (2000).
[11] C. Letellier and G. Gouesbet, J. Phys. IV **6**, 1615 (1996).
[12] Y. Mao, W. K. S. Tang, and M.-F. Danca, Appl. Math. Comput. (submitted).
[13] A. M. Stuart and A. R. Humphries, *Dynamical Systems and Numerical Analysis* (Cambridge University Press, Cambridge, 1996), Chap. 4.
[14] O. E. Rössler, "Chaos, hyperchaos and double-perspective," in *The Chaos Avant-Garde: Memories of the Early Days of Chaos Theory*, edited by R. Abraham and Y. Ueda (World Scientific, Singapore, 2000).
[15] J. K. Hale and N. M. Stavrakakis, Appl. Anal. **26**, 271 (1998).
[16] M. W. Hirsch and C. C. Pugh, "Stable manifolds and hyperbolic sets," in *Proceedings of the Symposium on Pure Mathematics 14* (American Mathematical Society, New York, 1970), pp. 133–164.
[17] M. W. Hirsch, S. Smale, and R. L. Devaney, *Differential Equations, Dynamical Systems and An Introduction to Chaos*, 2nd ed. (Elsevier, New York, 2004).
[18] J. Milnor, Commun. Math. Phys. **99**, 177 (1985).
[19] A. Segatti, Discrete Contin. Dyn. Syst. **14**, 801 (2006).
[20] G. Chen and T. Ueta, Int. J. Bifurcation Chaos Appl. Sci. Eng. **9**, 1465 (1999).
[21] X. Luo, M. Small, M.-F. Danca, and G. Chen, Int. J. Bifurcation Chaos Appl. Sci. Eng. **17**, 3235 (2007).