On a class of non-smooth dynamical systems: a sufficient condition for smooth vs nonsmooth solutions

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Abstract

In this paper we present a possible classification of the elements of a class of dynamical systems, whose underlying mathematical models contain non-smooth components. For this purpose a sufficient condition is introduced. To illustrate and motivate this classification, three nontrivial and realistic examples are considered.

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1 Introduction

Non-smooth time-continuous[1] dynamical systems (d.s.) appear in a large number of problems from mechanics (dry friction with stick and slip modes, impacts, oscillating systems with viscous damping, elasto-plasticity), electrical engineering (electrical circuits and networks with switches, power electronics) theory of automatic and optimal control, games theory, walking machines, biological and physiological systems and everywhere non-smooth characteristics are used to represent switches. In other words, in the real word non-smoothness is common.
The underlying mathematical models can be described by a set of first-order differential equations with non-smooth components. In particular, non-smoothness is due to the discontinuity of the state variable, of the associated vector field, of Jacobian (partial derivatives) or higher order discontinuity. Despite the fact that the terms "discontinuous" or "non-smooth" could be sometimes synonymous, however significant differences between underlying behavior, numerical solutions etc. may appear. The purpose of this paper is quite to classify precisely the models of a such class of d.s.

Our classification can be useful by several reasons. As example a 'non-smooth' d.s. which is continuous can be modelled using classical numerical methods, while if it is discontinuous special numerical methods should be used.

The known Filippov systems [1], discontinuous with respect to the state variable, represent a subclass of these systems.

In the practical examples the non-smoothness appears because of switch like functions (piecewise smooth functions, see Appendix) which can be continuous or piecewise continuous (e.g. signum, absolute value, Heaviside function-also known as the "unit step function", maximum etc.).

The behavior of non-smooth d.s. is much more complex than that of smooth d.s. and much research effort in applied science and engineering has focussed on non-smooth d.s. Let us recall some reasons: the numerical integration of the underlying non-smooth initial value problem (i.v.p.) is a difficult task, especially for the discontinuous d.s. where only special difference methods can be used (see e.g. [2, 3]); the standard methods of smooth d.s. theory rely heavily on linearization, while non-smooth d.s. in

[1] In this paper the functions are considered time-continuous, the continuity property being considered with respect to the state variable.

general does not have a linearization; in comparison with smooth d.s., the non-smooth systems can have trajectories which collide with some discontinuity surfaces in phase space (thus, new kind of bifurcations phenomena could arise [4]); the i.v.p. may have not any solutions, this situation could be, generally overdone by using the tools of differential inclusions [5].

However, since the applications and experimental results appear in many domains, the research certainly worth the effort.

In this paper we focus on d.s. which can be modelled by the following autonomous i.v.p.

$$\dot{x} = f(x), \quad x(0) = x_0, \quad x_0 \in \mathbb{R}^n, \quad t \in I = [0, \infty), \tag{1}$$

under the standing assumption that $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded. We emphasize too that f is not required to be continuous and has the following form

$$f(x) = g(x) + h(x), \tag{2}$$

with $g : \mathbb{R}^n \to \mathbb{R}^n$ vector function. As in the great majority of the practical examples, we can make the following assumptions: h is a vector function $h : \mathbb{R}^n \to \mathbb{R}^n$ with the following form

$$h(x) = \begin{pmatrix} \sum_{i=1}^{m_1} h_i^1(x_i) u_i^1(x), \\ \vdots \\ \sum_{i=1}^{m_n} h_i^n(x_i) u_i^n(x), \end{pmatrix},$$

where the scalar components $x_i \longmapsto h_i^j(x_i) \in \mathbb{R}$ are piecewise smooth functions with a single non-smoothness point $\alpha_i^j \in \mathbb{R}$. g and the real functions $u_i^j : \mathbb{R}^n \to \mathbb{R}$ (at least one of them being non-vanished) are smooths[1], for $j = 1, 2, \ldots n$, $i = 1, 2, ..., m_j$, m_j being some positive integers.
Hereafter the index i and j are considered to be $j = 1, 2, \ldots n$, $i = 1, 2, ..., m_j$ unless otherwise stated.

Function of u properties i.v.p.(1) may model a continuous, smooth, discontinuous or non-smooth d.s.

[1]In practical exemples very few cases with u_i^j continuous non-smooth were encountered. For this reason we considered mext u_i^j smooth. However, the results remain the same.

Notation 1 *(i) The non-smoothness null set of* h *will be denoted by* $M = \{\overline{x}_i^j = \left(x_1, \ldots, x_{i-1}, \alpha_i^j, x_{i+1}, \ldots, x_n\right) \in \mathbb{R}^n\}$*, and the smoothness domain by* $D = \mathbb{R}^n \backslash M$*.* D *consists of finite and open unidimensional domains* $D_i^{\pm\ j} \subset \mathbb{R}$*. Their boundaries are smooth hyperplanes,* Σ*, which have the equation* $x_i = \alpha_i^j$*. Thus* $D_i^{-j} = \{x_i \in \mathbb{R} |\ x_i < \alpha_i^j\}$ *and* $D_i^{+j} = \{x_i \in \mathbb{R} |\ x_i > \alpha_i^j\}$*. On the boundary of* $D_i^{\pm\ j}$ *the derivatives of* h_i^j *make switch.*
(ii) Let denote by $\mathcal{H}$ *the class of all functions* h_i^j *piecewise smooth and possibly continuous on* $\mathbb{R}$*.*

Remark 2 *(i) Hereafter, in this paper, by non-smooth function one understand a continuous/discontinuous piecewise smooth function.*
(ii) d.s. modeled by i.v.p.(1) shall be considered smooth only if its right-hand side f *is smooth. The only presence of the* h_i^j *functions does not represents a sufficient condition for* f *non-smoothness or even discontinuity. For this reason, the term "non-smooth" d.s. used without some additional assumptions in these cases may be improper.*

For $u_i^j = const$ and $h_i^j(x_i) = sgn(x_i)$ we get the particular case of the Filippov systems [1]. A particular class of d.s. modeled by i.v.p.(1) was treated in [6].

The structure of this paper is the following: Section 2 introduces the main property concerning the continuity and smoothness of the right-hand side of i.v.p.(1), while Section 3 introduces the classification of the d.s. modeled by (1) beside practical examples, using the result in Section 1.

2 Continuity and smoothness of the right-hand side f

Function of u_i^j properties we have the following main result concerning the right-hand side f of i.v.p.(1)

Proposition 3 *Suppose* f *defined by* (2) *with* u_i^j *and* g *smooth and* $h_i^j \in \mathcal{H}$
(i) If at least one of the following assumptions is true:
i1) h_i^j *are continuous;*
i2) u_i^j *verifies the following condition*

$$u_i^j(\overline{x}_i^j) = 0, \tag{*}$$

then f *is continuous on* $\mathbb{R}^n$;
(ii) If u_i^j *verifies (*) and moreover*

$$\frac{\partial\ u_i^j}{\partial\ x_i}(\overline{x}_i^j) = 0, \tag{**}$$

then f *is smooth on* $\mathbb{R}^n$.

Proof. (i) The proof for the case h_i^j continuous is obvious.
Let us consider (*) verified. The continuity of f on D follows from the continuity of g, u_i^j and (piecewise) continuity of h_i^j.
Taking account that the state variable component x_i belongs either to some $D_i^{+\ j}$ or $D_i^{-\ j}$, on $\overline{x}_i^j \in M$ each component $h_i^j u_i^j$ verifies

$$\lim_{\substack{x \to \overline{x}_i^j \\ x \in D}} h_i^j(x_i) u_i^j(x) = \lim_{\substack{x_i \to \alpha_i^j \\ x_i \in D_i^{\pm\ j}}} h_i^j(x_i) \lim_{\substack{x \to \overline{x}_i^j \\ x \in D}} u_i^j(x) = 0,$$

due to the boundness of h_i^j on $D_i^{\pm\ j}$ and because from (*) we have, following the u_i^j continuity on $\mathbb{R}^n$

$$\lim_{\substack{x \to \overline{x}_i^j \\ x \in D}} u_i^j(x) = u_i^j(\overline{x}_i^j) = 0.$$

Consequently f is continuous in $\overline{x}_i^j$ since, using the continuity of g in these points and (*), one obtains

$$\lim_{\substack{x \to \overline{x}_i^j \\ x \in D}} f^{\ j}(x) = \lim_{\substack{x \to \overline{x}_i^j \\ x \in D}} g^j(x) = g^j(\overline{x}_i^j) = f(\overline{x}_i^j),$$

which proves that f is continuity on $\mathbb{R}^n$.

(ii) On D, f is smooth due to the smoothness of g, u_i^j and the piecewise smoothness of h_i^j.
On D the partial derivatives of h are

$$\frac{\partial\ h^j}{\partial\ x_k}(x) = \sum_{i=1}^{m_j} \left[\frac{d\ h_i^j(x_i)}{d\ x_k} u_i^j(x) + h_i^j(x_i) \frac{\partial\ u_i^j}{\partial\ x_k}(x) \right],\ \ k = 1, ..., n.$$

On $\overline{x}_i^j \in M$, (*) and (**) yield

$$\lim_{\substack{x \to \overline{x}_i^j \\ x \in D}} \frac{\partial\ h^j}{\partial\ x_k}(x) = 0, \quad k = 1, ..., n,$$

and the partial derivatives of f verifies

$$\lim_{\substack{x\to\overline{x}_i^j\\ x\in D}}\frac{\partial\ f^{\ j}}{\partial\ x_k}(x)=\lim_{\substack{x\to\overline{x}_i^j\\ x\in D}}\frac{\partial\ g^j}{\partial\ x_k}(x)=\frac{\partial\ g^j}{\partial\ x_k}(\overline{x}_i^j)=\frac{\partial\ f^{\ j}}{\partial\ x_k}(\overline{x}_i^j).$$

Thus, the partial derivatives of f exist and are continuous in $\overline{x}_i^j$. Therefore f is smooth on $\overline{x}_i^j$ and on $\mathbb{R}^n$. This ends the proof. ■

Remark 4 *Conditions (*) and (**) are very restrictive, therefore only few particular cases of i.v.p.(1) with functions verifying (*) or/and (**) were encountered. The most used example of such functions are the polynomials which can be decomposed as follows* $u(x)=(x_i-\alpha_i)^m\,\overline{u}(x)$, $x\in\mathbb{R}^n$, *where* $\overline{u}$ *is a real polynomial, with* $m=1$ *for (*), and* $m\geq 2$ *for (**). Also trigonometric functions can be found in practical examples.*

For the case $u(x)=const$ in [7] a generalized derivative for f was introduced.

If h_i^j are discontinuous, special precaution should be taken, since in α_i^j i.v.p.(1) may have not sense. Therefore a common device to obtain a precise mathematical setup is to replace these functions h_i^j by their multivalued version which can be handled via differential inclusions theory (see e.g. [1] or [5]). For instance, for sgn function, its multivalued form is

$$Sgn(x)=\begin{cases}\{-1\}, & x<0,\\ [-1,1]\,, & x=0,\\ \{+1\}, & x>0,\end{cases}$$

and the discontinuous i.v.p. $\dot{x}=sgn(x)$, $x(0)=x_0$, transforms into the set-valued i.v.p.(differential inclusion) $\dot{x}\in Sgn(x)$, $x(0)=x_0$ which, due to the regularity of the set-valued function Sgn, has several generalized (Filippov) solutions (see e.g. [6]). This is one of the major differences between discontinuous i.v.p., where the solutions as we have seen imply special approach, and continuous non-smooth i.v.p. where, due the continuity, the existence is assured (Péano's theorem).

3 Dynamical systems modeled by i.v.p.(1)

We consider next the d.s. modeled by (1) as being continuous, discontinuous, smooth or non-smooth if the right-hand side enjoy the mentioned properties.

Using Proposition 3 we can formulate the main result which allows to classify d.s. modeled by i.v.p.(1), function of his right-hand side properties.

Theorem 5 *Let i.v.p.(1) with* u, g *smooth and* $h_i^j \in \mathcal{H}$

1. *If* u *verifies (*) then the i.v.p. defines a continuous non-smooth d.s.*
2. *If* u *verifies (*) and (**), then the i.v.p. defines a smooth d.s.*

All the possible cases treated by the above theorem are schematically depicted in Fig. 1.

Remark 6 *In practical examples there are i.v.p.(1) with non-smooth functions* u *(see Example 8), which can be classified too by the above algorithm.*

Next let us consider the following examples from electrical circuits, mechanics with dry friction and biology, without initial conditions. The chaotic behaviors of these systems are not studied here, the details on their dynamics could be found in the mentioned references. The phase portraits and time series were plotted using a program code which implemented a special numerical method for differential inclusions, corresponding to the underlying discontinuous d.s.

Example 7 *Models of some electrical circuits can be accurately implemented with resistors, capacitors, diodes and amplifiers. The following example is a modified mathematical variant of the known Chen system [8]* [2]

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1), \\ \dot{x}_2 &= cx_2 + (c - a - x_3)\ x_1^p sgn(x_1), \\ \dot{x}_3 &= -bx_3 + x_1 x_2^p\ sgn(x_2), \quad a, b, c \in \mathbb{R}^+, \end{aligned} \tag{3}$$

where we considered the cases $p \in \{0, 1, 2\}$. *Here*

$$g(x) = \begin{pmatrix} a(x_2 - x_1) \\ cx_2 \\ -bx_3 \end{pmatrix},$$
$$h(x) = \begin{pmatrix} 0 \\ (c - a - x_3)x_1^p sgn(x_1) \\ x_1 x_2^p\ sgn(x_2) \end{pmatrix}.$$

[2] The circuitry is practically realizable (see e.g. [9] where details on the electrical circuit implementations are presented).

The functions u_i^j *are variables*

$$u_1^2(x) = (c - a - x_3)x_1^p,\ u_2^3(x) = x_1 x_2^p,\ u_i^j(x) = 0, \quad for \ \ i \neq 1,2 \ \ and \ \ i \neq 2,3,$$
$$h_1^2 = sgn(x_1), \quad h_2^3 = sgn(x_2).$$

(a) For $p = 0$ *and* $a \neq c$ *one obtains the discontinuous, piecewise linear form of another variant of the Chen's system studied in [10]. The dynamics for* $a = 1.18,\ b = 0.168,\ c = 7$ *and* $q = 0.1$ *is chaotic (Fig.2). For* $a = c = 1,\ b = 0.16,$ *and* $q = 0.9,$ *the d.s. is continuous non-smooth (Fig.3).*
(b) For $p = 1$ *and* $a = c,$ *the condition (*) is fulfilled. Therefore the right-hand side is continuous, non-smooth, the considered d.s. being a continuous non-smooth one, while for* $a \neq c$ *is a discontinuous one.*
(c) For $p = 2$ *and* $a = c$ *the right-hand side is smooth since* u_i^j *verify both (*) and (**). In this case the system is smooth. For* $a \neq c$ *the right-hand side is discontinuous and the underlying d.s. is discontinuous. No chaotic behavior was found for* $p = 1$ *and* $p = 2$*, except regular motions (Fig.4).*

Example 8 *The next example is a nonautonomous friction oscillator periodically excited by a forcing and is governed by the following equation ([11, pp 225])*

$$\ddot{x} + x + a\left[\mu(1) + \mu(\dot{x} - 1)sgn(\dot{x} - 1)\right] = \gamma\cos(\omega t),$$

with the friction characteristic $\mu(y)$

$$\mu(y) = \frac{\mu_0 - \mu_1}{1 + \lambda_0\,|y|} + \mu_1 + \lambda_1\,|y|^2\,,$$

where $a = 10,\ \mu_0 = 0.4,\ \mu_1 = 01,\ \lambda_0 = 1.42,\ \lambda_1 = 0.01,\ \ \gamma = 0.7$ *and the control parameter* $\omega \in [1, 1.15]$*. The standard autonomous form is*

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 - a\left[\mu(1) + \mu(\dot{x} - 1)sgn(\dot{x} - 1)\right] + \gamma\cos(x_3), \\ \dot{x}_3 &= \omega, \end{aligned} \tag{4}$$

with

$$g(x) = \left(\begin{array}{c} x_2 \\ -x_1 - a\mu(1) \\ \omega \end{array} \right), \ h(x) = \left(\begin{array}{c} 0 \\ a\mu(x_2 - 1)sgn(x_2 - 1) \\ 0 \end{array} \right).$$

Here $u_1^2(x) = \mu(x_2 - 1)$ *and* $u_i^j(x) = 0$ *for* $j \neq 2$, $i \neq 1$. $h_1^2(x_1) = sgn(x_2 - 1)$ *is discontinuous of* $\mathcal{H}$ *class, while* u *is continuous on* $\mathbb{R}$, *piecewise smooth and does not verifies neither (*) nor (**) in* $\overline{x}_1^2 = (x_1, \alpha_1^2, x_3) = (x_1, 1, \omega)$ *(it is easy to see that* $\mu(y) > 0 \ \ \forall y \in \mathbb{R}$, *see Fig.5 and Remark 6). Therefore the right-hand side of (4) is non-smooth and the d.s. is discontinuous. The "grazing" phenomenon (tangential contact between trajectory and the discontinuity surface [11]) for the dry friction can be seen in the Fig.6 where a chaotic trajectory is depicted.*
Several examples of dry friction problems can be found in [11].

Example 9 *The last considered example is a simple one taken from biology [12] describing the distribution of predators among two different habitat patches*

$$\begin{aligned} \dot{x}_1 = & \ ax_1 - v_1 x_1 x_3, \\ \dot{x}_2 = & \ x_2 - v_2 x_2 x_3, \\ \dot{x}_3 = & \ v_1 x_1 x_3 + v_2 x_2 x_3 - x_3, \ a > 0, \end{aligned} \tag{5}$$

where the control $v(t) = v\left(v_1(t), v_2(t)\right)$ *is a measurable function. In behavioral ecology it is often assumed that each individual behaves optimally, i.e. the control* $v = (v_1, v_2)$ *has to follow the optimallity constraint (see [12] for details)*

$$v \in \begin{cases} (1,0) & if \ \ x_1 > x_2, \\ (0,1) & if \ x_1 < x_2, \\ \{(v_1, v_2) \, | v_1 + v_1 = 1, \ v_i \geq 0\} & if \ \ x_1 = x_2. \end{cases}$$

Thus, the discontinuous i.v.p.(5) transformed into a differential inclusion, or multivalued problem. A chaotic behavior is presented in Fig.7.

4 Conclusion

In this paper the classification of d.s. modeled by i.v.p.(1) was achieved using the properties of the right-hand side function f. Significant practical examples were analyzed and classified via the proposed scheme.

Using this algorithm, we proved that despite the presence of non-smooth components in the underlying i.v.p. the d.s. may be continuous or even smooth.

Classes of d.s. as piecewise linear, discontinuous of Filippov like type, are included, following the presented scheme, into the general class of d.s. (1).

The most encountered practical examples of systems modeled by i.v.p.(1) are, as expected, discontinuous.

It would be interesting to see if requirements on initial conditions could assure fulfillment of the requirement of the behavior of the solution on M.

A study on the differences between the dynamics of these subclasses of systems modeled by i.v.p.(1) remains a task for a future work.

Appendix

Definition 10 *A smooth function is a function that has continuous derivatives up to some desired order over some domain (i.e. a function of* $C^m, m \geq 1$ *class).*

Definition 11 *A* piecewise function *is a function that is defined on a sequence of intervals.*

A common example is the absolute value $|\bullet|$.

Definition 12 *A function is* piecewise continuous *if it's domain can be partitioned into a sequence of finite number of intervals such that it is continuous-over each such interval, and there is a finite distance between each pair of break points.*

The $sign$ is a classical example of piecewise continuous function.

Definition 13 *A function* f *is* piecewise smooth *on an interval if both f and his partial derivatives of* $m \geq 1$ *order, are piecewise continuous on the interval*

The definitions of piecewise continuous, piecewise differentiable and so on are therefore made, to require that the 'pieces' of the function are continuous (respectively differentiable), but that at the end points failure of those conditions is allowed.
An one-dimensional example is the piecewise smooth sawtooth function $f(x) = x,\ x \in [-\pi,\ \pi]$, extended periodically on the real line; this function is discontinuous at $x = (2n+1)\pi$ for all integer values of n.
Also the $sign$ function is a piecewise smooth discontinuous function, while the absolute value function is piecewise smooth continuous.

Remark 14 *A piecewise smooth functions in I, is not necessary continuous at every point in I, but is only allowed to have a finite number of jump discontinuity.*

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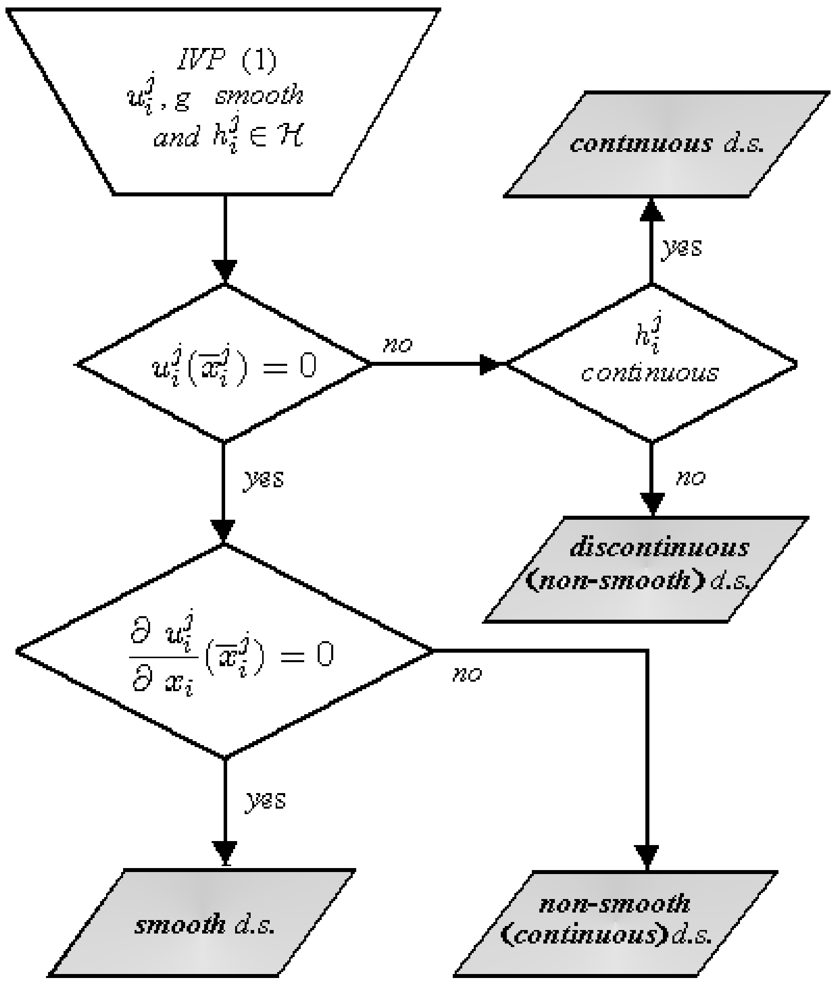

Figure 1: Classification of the dynamical systems modeled by (1).

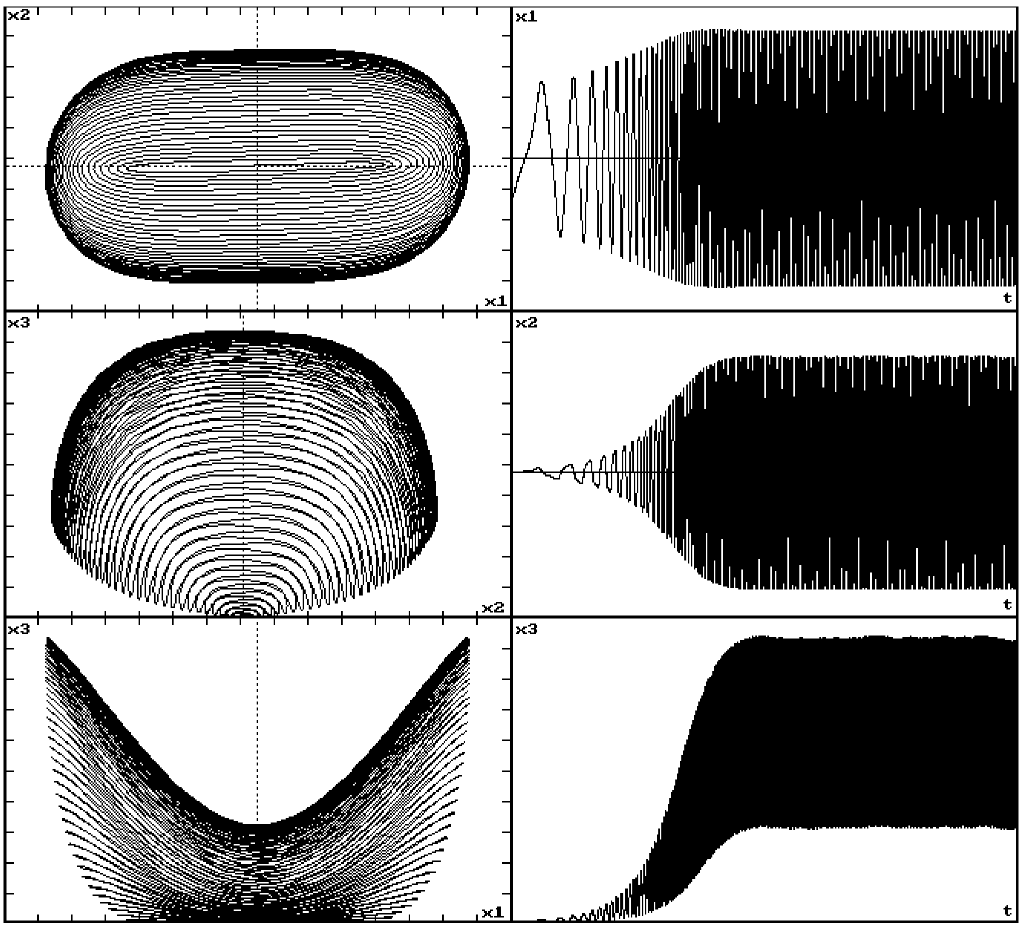

Figure 2: A chaotic trajectory of the discontinuous d.s. (3) with $p = 0$.

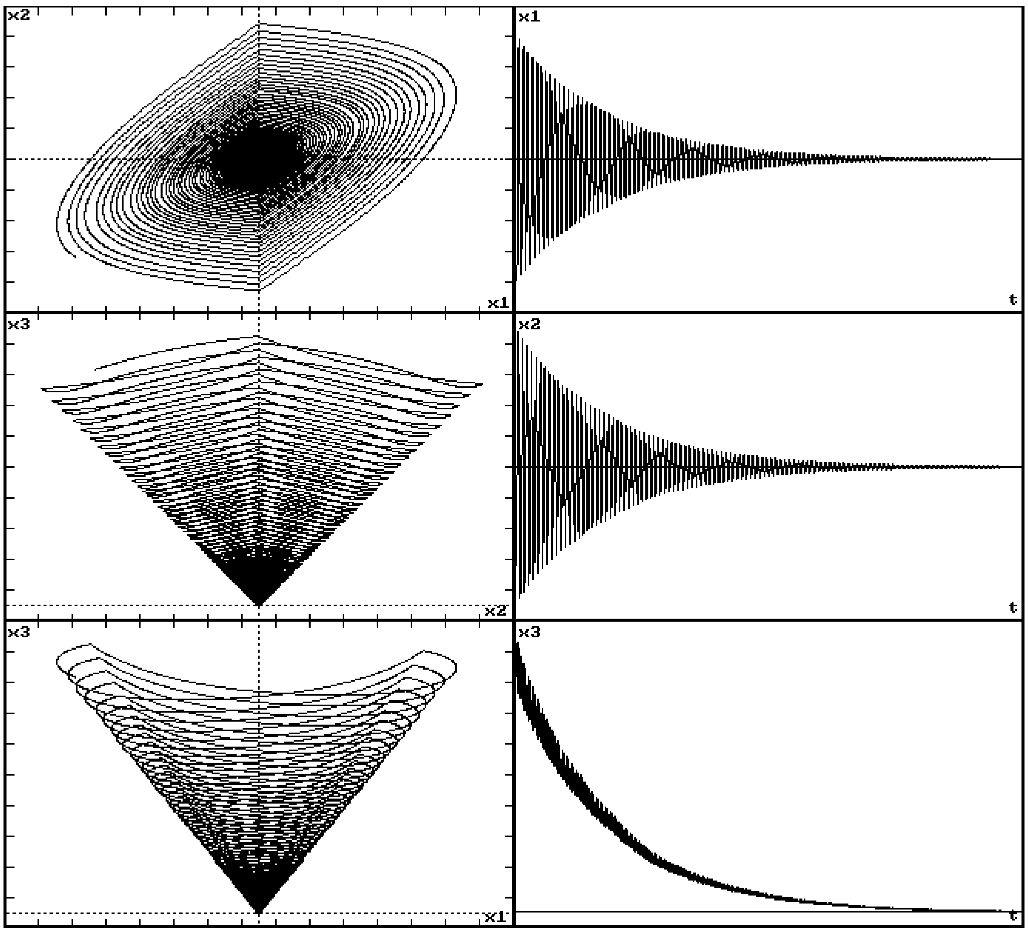

Figure 3: Attractive fixed point of (3) for $a = c$.

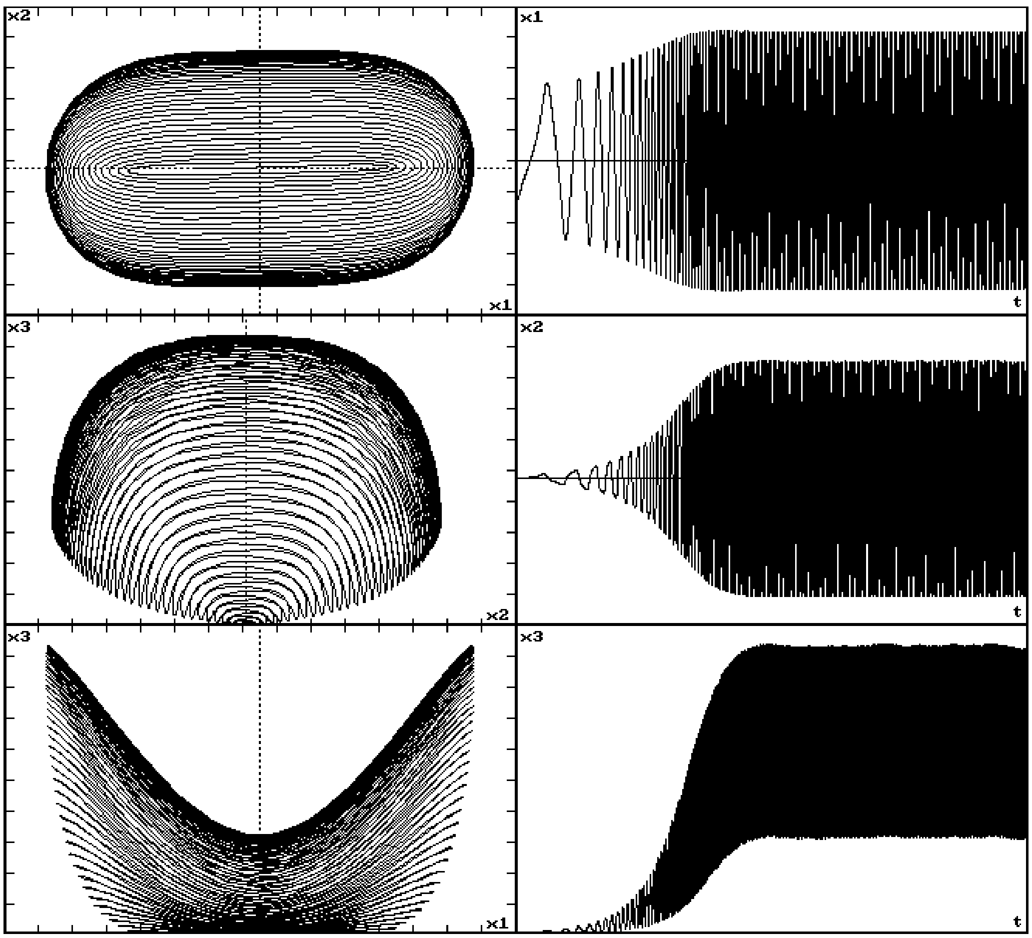

Figure 4: A stable limit cycle of the d.s. (3).

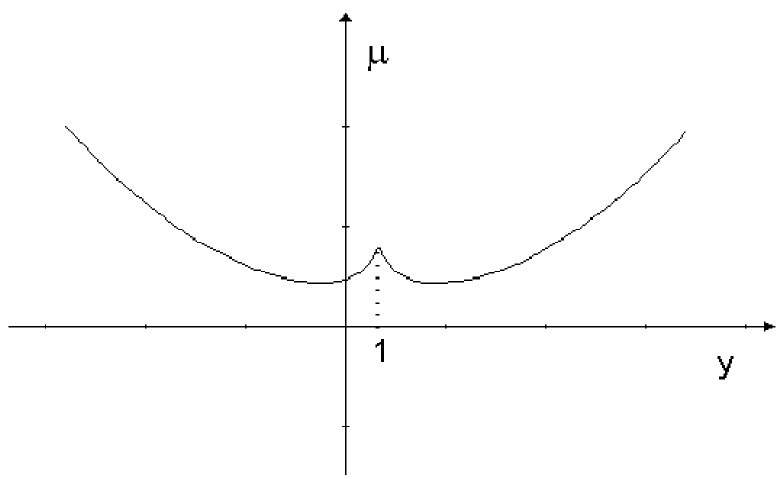

Figure 5: The graph of the function u_1^2 from the model (4).

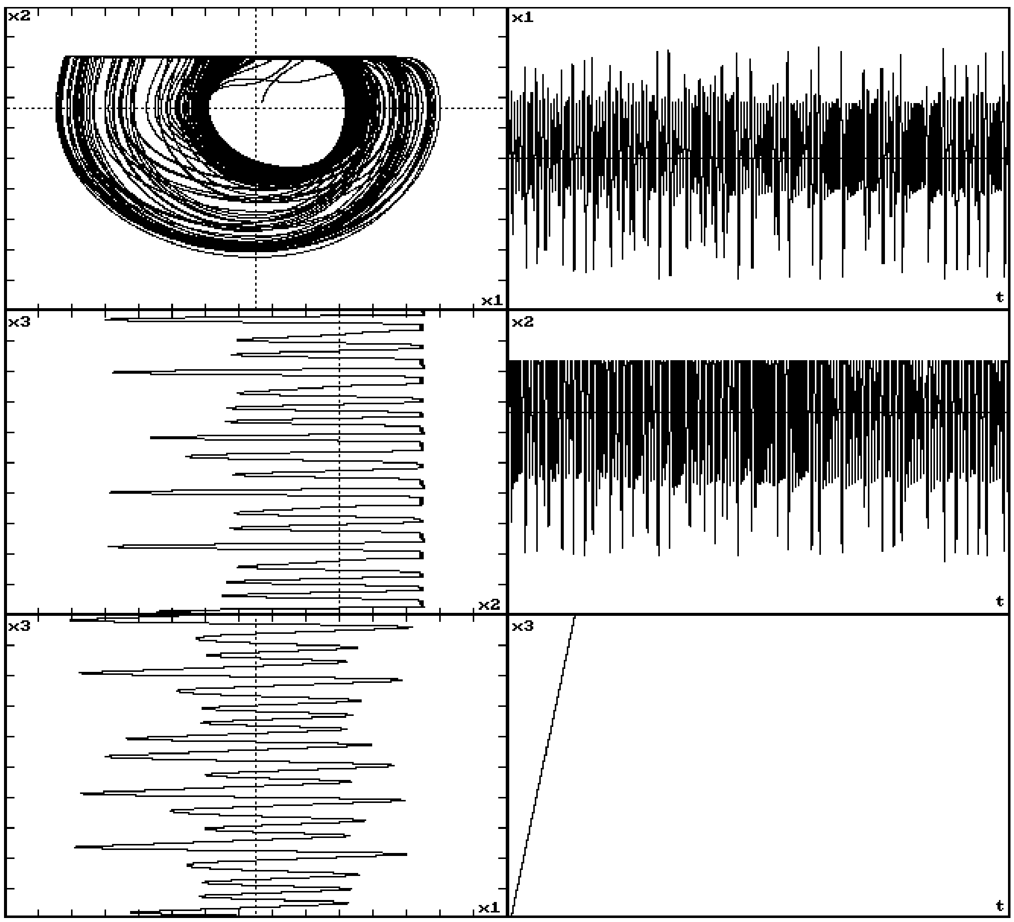

Figure 6: Grazing phenomen for the d.s. (4).

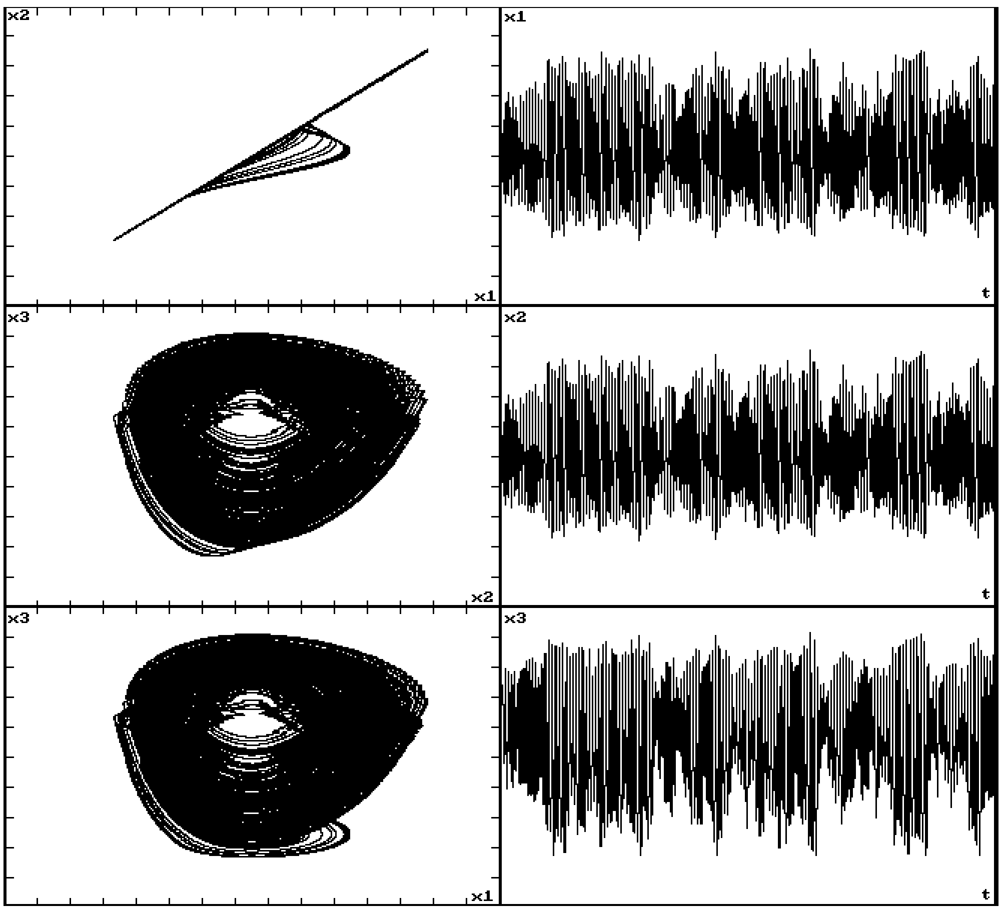

Figure 7: A chaotic trajectory of the d.s. (5).