Piecewise linear Chen system of fractional-order — Approximating stable attractors by parameter switching

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This article is dedicated to Professor Guanrong(Ron) Chen on the occasion of his 65th birthday

Abstract

In this paper we present a new variant of Chen's system — a piecewise linear Chen system of fractional-order. The discontinuous system is transformed via Filippov's regularization and using Celinna's Theorem, into a continuous system. By numerical simulations, we reveal chaotic behavior and also the existence of small parameter windows where, for some fixed bifurcation parameter and depending on initial conditions, coexistence of stable attractors and chaotic attractors is possible. Using an algorithm to switch the bifurcation parameter, every stable attractor can be numerically approximated.

Keywords: PWL Chen attractor of fractional-order; parameter switching; Cellina's Theorem, Filippov regularization; Sigmoid function; Bifurcation diagram

1 Introduction

There are several paradigmatic three dimensional chaotic flows. Aside from the ubiquitous Lorenz and Rössler systems, one of the most intricate and widely studied is the system of Chen proposed in 1999 [?]. Each of these systems represent a topological distinct genus of chaos. In 2002 Aziz-Alaoui and Chen [?] presented a Piecewise Linear (PWL) Chen system modeled by the following Initial Value problem (IVP)

$$\begin{aligned}
&\dot{x}_1 = a\left(x_2 - x_1\right), \\
&\dot{x}_2 = \left(c - a - x_3\right) sgn(x_1) + cdx_2, \quad x(0) = x_0, \quad t \in [0, T], \\
&\dot{x}_3 = x_1 sgn(x_2) - bx_3,
\end{aligned} \tag{1}$$

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with $x_0 \in \mathbb{R}^3$, $T > 0$, a, b, c, d some positive real parameters verifying the condition $a < c$. Whereas the origin Chen system [?] is described by

$$\begin{array}{ll} \dot{x}_1 = a\left(x_2 - x_1\right), & \\ \dot{x}_2 = \left(c - a\right)x_1 - x_1 x_3 + c x_2, & x(0) = x_0 \quad t \in [0, T], \\ \dot{x}_3 = x_1 x_2 - b x_3, & \end{array} \tag{2}$$

(with $a = 35$, $b = 3$, and $c = 28$) this new system (**??**), is a piecewise continuous variant. A detailed asymptotic analysis of this new system, including a study of the chaotic behavior with the bifurcation parameter, is presented in [?].

In this paper we present a new and more general extension of the PWL system (**??**): the PWL Chen's system of fractional-order. Define the fractional differential system

$$\begin{array}{ll} D_*^{q_1} x_1 = a\left(x_2 - x_1\right), & \\ D_*^{q_2} x_2 = \left(c - a - x_3\right) sgn(x_1) + c d x_2, & x(0) = x_0, \quad t \in [0, T], \\ D_*^{q_3} x_3 = x_1 sgn(x_2) - b x_3, & \end{array} \tag{3}$$

where D_*^q (for $q = (q_1, q_2, q_3)$) stands for Caputo's differential operator of order q with starting point 0 [?, ?, ?]. Recall that the Caputo differential operator is a fractional extension of differentiation defined for $q \in \mathbb{R}$ by

$$D_*^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q+1-n}} d\tau.$$

This definition employs a generalization of an integer order derivative n, Cauchy's identity expressing n-repeated integration with a single integral, and Euler's Γ function. In contrast to alternative definitions of fractional differentiation, Caputo's derivative gives a physical interpretation to the included initial conditions necessary in practical problems [?, ?]. Using this definition, we avoid the expression of initial conditions with fractional derivatives and the initial condition in (**??**) can be considered in the standard form $x(0) = x_0$. In most practical examples, one takes $q_i \leq 1$, $i = 1, 2, 3$.

With these definitions in place it is straightforward to check that for $q = (1, 1, 1)$, the system (**??**) reduces to the ordinary integer-order PWL Chen system (**??**). In the remainder of this paper we explore this system — both for the integer order derivatives and for fractional-order extensions.

Let us denote by p the single scalar bifurcation parameter, which for the considered system (**??**) can be any of the four parameters a, b, c or d. Two representative cases have been considered here: $p := d$ integer case $q = (1, 1, 1)$, and $p := a$ integer case $q = (1, 1, 1)$, and the fractional-order case $q = (0.99, 0.98, 0.97)$. For the sake of conciseness we do not present results of computation here, but we have found that the other two cases ($p := b$ and $p := c$) demonstrate similar behaviour. Aided by extensive numerical computations and computer simulations, we studied the bifurcation behaviour of each of these parameters sets. We find a particularly rich bifurcation behaviour, as demonstrated by the exemplar Bifurcation Diagrams (BD) in Fig. **??**. The results are for: (a) $p := d$ during the integer-order case, with $a = 1.15$; (b) $p := a$ in the integer order case; and (c) for $p := a$ with the fractional-order case. In each case we set $b = 0.15$ and $c = 2$.

It is easy to verify that, in addition to the origin, the system has four equilibrium points $X^*\left(\pm\frac{ab-bc}{bcd\pm1}, \pm\frac{ab-bc}{bcd\pm1}, \pm\frac{a-c}{bcd\pm1}\right)$. The sign of the first two coordinates can be found in Table **??**. The coordinates present symmetries which are also evident in the simulations of the state variables x_1 and x_2. These symmetries come from the invariance of the vector field under transform $(x, y, z) \rightarrow (-x, -y, z)$. To note that for the chosen values for parameters a, b, c and d, with $a < c$, the state coordinate x_3^* is positive while $x_{1,2}^*$ can be either positive and negative.

$sgn(x_1)$	$sgn(x_2)$	$X^*(x_1^*, x_2^*, x_3^*)$
$-$	$-$	$X_1^*\left(-\frac{ba-bc}{bcd-1}, -\frac{ba-bc}{bcd-1}, \frac{a-c}{bcd-1}\right)$
$+$	$-$	$X_2^*\left(\frac{ba-bc}{bcd+1}, \frac{ba-bc}{bcd+1}, -\frac{a-c}{bcd+1}\right)$
$-$	$+$	$X_3^*\left(-\frac{ba-bc}{bcd+1}, -\frac{ba-bc}{bcd+1}, -\frac{a-c}{bcd+1}\right)$
$+$	$+$	$X_4^*\left(\frac{ba-bc}{bcd-1}, \frac{ba-bc}{bcd-1}, \frac{a-c}{bcd-1}\right)$

Table 1: Equilibrium points of PWL Chen's system.

As can be seen from the detail in Figure **??** (a) and (b), for $p := d$ and $p := a$ respectively, a new characteristic has been uncovered for this system. We see evidence for the coexistence of stable and chaotic motions in some parameter windows (details D_1 in Figure **??** (a) and D_2 in Figure **??** (b)). Therefore, for a specific value p in these windows, depending on the basin of attraction and initial conditions, we can find coexisting attractors. For example, for $a = 1.15$, $b = 0.15$, $c = 0.1$, and with $p := d = 0.361$ chosen in the window $[0.345, 0.370]$, we found two different attractors: a chaotic one and a stable cycle (see D_1 in Figure **??** (a)). For $b = 0.15$, $c = 2$, $d := 0.1$ and with $p := a$ chosen in the window $p \in [0.7, 0.8]$, the two attractors are plotted in the detail D_2 in Figure **??** (b).

In the remainder of this paper we show numerically, aided by computer simulations, that every stable attractor of Chen's system (**??**) can be numerically approximated by a Parameter Switching (PS) algorithm [**?**]. The PS algorithm switches p within a chosen set, while the IVP is numerically integrated with some numerical scheme with fixed step size. The convergence criteria for the PS algorithm to some desired attractor is presented in [**?**, **?**]. In what follows, the numerical integrations have been realized with the standard Runge-Kutta (RK4) method for the integer-order case and Adams-Bashforth-Moulton (ABM) method [**?**] for the fractional-order case (see also [**?**]).

The paper is structured as follows: in Section **??** the PS algorithm is explained, Section **??** presents the continuous approximation of the IVP (**??**) and in Section **??** several stable attractors of PWL Chen's system of fractional-order are approximated with the PS algorithm.

2 Parameter switching algorithm

Let a continuous system of fractional-order be modeled by the following IVP

$$D_*^q x = f(x) + pAx, \quad t \in I = [0, T], \quad x(0) = x_0, \tag{4}$$

where: $A \in \mathbb{R}^{n \times n}$ is a real square constant matrix; $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear, at least continuous, function and, $q \leq 1$. The great majority of known systems of integer or fractional-order, including the usual Lorenz, Chen, Rössler, Chua, Rikitake (and many others), are modeled by this IVP. In [**?**, **?**] it was proved analytically and verified numerically that the PS algorithm allows for the numerical approximation of any desired attractor by switching the control parameter within a chosen finite set of values while the underlying IVP is numerically integrated with some fixed step size numerical method.

Next, assume that the IVP enjoys the uniqueness (Lipschitz continuity is a common sufficient conditions for both integer and fractional case (see [**?**] Corollary 6.9 for FDE uniqueness)). Furthermore, denote by $\mathcal{P}_N = \{p_1, p_2, ..., p_N\} \subset \mathbb{R}, \quad N \geq 2$, the switching values set.

Remark 1. *Due to this uniqueness assumption, it is reasonable to consider that for each $p_i \in \mathcal{P}_N$, $i \in \{1, 2, ..., n\}$ there is a corresponding unique attractor A_{p_i}. We follow this assumption in this paper — as is usual for numerical tests — and refer to this as* the *trajectory of the underlying numerical solution, after sufficiently long transients are removed.*

By choosing $\mathcal{P}_N$, while the underlying IVP is numerically integrated with a fixed step-size h (in this paper, we use a fourth order Runge-Kutta integrator for the integer case and the ABM method for the fractional case), PS switches in some deterministic (periodic) way the control parameter within $\mathcal{P}_N$, for relative short time subintervals. The obtained "switched solution" will approximate the "averaged solution" obtained for p replaced with the average of switched values. Following the Remark **??**, the attractor corresponding to the switched solution, will approximate as closely as desired (depending on the numerical integration accuracy limitation), the attractor corresponding to the averaged solution.

Schematically, for a chosen set $\mathcal{P}_N$ with the "weights" set $\mathcal{M} = \{m_1, m_2, ..., m_N\}$, and a fixed step-size h, the way in which PS algorithm works can be expressed schematically as follows

$$[m_1 p_1, m_2 p_2, ..., m_N p_N], \tag{5}$$

means that for the first m_1 integration steps the control parameter p will take the value p_1, then, for the next m_2 integration steps, $p = p_2$, and so on until, for m_N times, $p = p_N$, after which, again $p = p_1$ for m_1 times, then $p = p_2$ for m_2 times and so on until the considered time integration interval, $[0, T]$, is covered. For example $[2p_1, p_2]$ means that while the IVP is integrated, p will take the values as follows: $p = p_1$, $p = p_1$, $p = p_2$, $p = p_1$, $p = p_1$, $p = p_2$, and so on.

Let us denote the "weighted average" of the values of $\mathcal{P}_N$ by

$$p^* := \frac{\sum_{i=1}^N p_i m_i}{\sum_{i=1}^N m_i}. \tag{6}$$

Then, the "switched attractor", A^*, obtained with the PS algorithm, will approximate the "averaged attractor", A_{p^*}, obtained by integration of the underlying IVP with p replaced with p^*. The analytical proof of the convergence, for the integer-order case, is presented in [?, ?], while for fractional-order systems the convergence has been computationally verified for several systems (see e.g. [?]).

This PS algorithm can be used both as a control and an anticontrol-like techniques, when, by some objective reasons, some certain targeted parameter value p^* cannot be accessed directly. In this case, we have to select $\mathcal{P}_N$, m_i, $i = 1, 2, ..., N$ and an adequate scheme (??) to obtain, with (??), the targeted value p^*.

The PS algorithm can help enrich our understanding of what happens in some real systems when the control parameter is switched by natural or imposed causes. However, while most known control/anticontrol algorithms "force" the trajectory to change its characteristics and behavior, the PS algorithm allows to obtain any desired existing attractor.

Finally, the PS algorithm proves that p switchings present a robustness-like property in the following sense: for whatever sets $\mathcal{P}_N$, $\mathcal{M}$, if $p_{min} = \min\{\mathcal{P}_N\}$ and $p_{max} = \max\{\mathcal{P}_N\}$, the obtained values p^* remains between p_{min} and p_{max}. Note that the PS algorithm applies also for discrete-time real systems [?] and some interesting results have been obtained for complex discrete-time systems [?] — although that is beyond what we consider here.

3 Continuous approximation of PWL Chen system

The IVP (??) is PWL. The main obstacle in applying the PS algorithm to PWL systems is the lack of numerical methods specifically devised for fractional differential equations (FDE) with discontinuous right-hand side (Filippov-like systems of fractional-order). This is one of the reasons that discontinuous systems of fractional-order have been not rigorously studied. One possible approach to surpass the obstacle of integration of discontinuous FDE, is to approximate continuously the underlying discontinuous initial value problem, according to the algorithm based on the transformation of the IVP into a set-valued one, which next can be continuously approximated with a single-value IVP [?]. Thus, piecewise constant components, like *sgn*, can be continuously approximated globally (on small neighborhoods of the graph) or locally (in small neighborhood centered in the discontinuity point $x = 0$). In this paper we use the global sigmoid approximation $\widetilde{sgn}$

$$\widetilde{sgn}(x) = \frac{2}{1 + e^{-\frac{x}{\delta}}} - 1, \tag{7}$$

which approximates globally the *sgn* function, on small neighborhoods of its graph (see Fig. ?? (a) where, for clarity, δ has be chosen larger ($\delta = 1e - 1$), while in Fig. ?? (b), *sgn* is drawn as a function of two variables x and δ). The parameter δ controls the slope of the sigmoid function, near the discontinuity $x = 0$. As proved in [?], an adequate choice for numerical purposes is δ in $\delta = 1e - 5$.

After approximation, In Chen's system (??), becomes

$$\begin{aligned} D_*^{q_1} x_1 &= a\left(x_2 - x_1\right), \\ D_*^{q_2} x_2 &= \left(c - a - x_3\right) \widetilde{sgn}(x_1) + c d x_2, \\ D_*^{q_3} x_3 &= x_1 \widetilde{sgn}(x_2) - b x_3. \end{aligned} \tag{8}$$

The cases studied in this paper are presented in Table ??

$p := d$	$p := a$
$\dot{x}_1 = a(x_2 - x_1)$, $\dot{x}_2 = (c - a - x_3)\widetilde{sgn}(x_1) + cpx_2$, $\dot{x}_3 = x_1\widetilde{sgn}(x_2) - bx_3$,	$D_*^{q_1}x_1 = p(x_2 - x_1)$, $D_*^{q_2}x_2 = (c - p - x_3)\widetilde{sgn}(x_1) + cdx_2$, $D_*^{q_3}x_3 = x_1\widetilde{sgn}(x_2) - bx_3$,
$\dot{x} = f(x) + pAx$	$D_*^q x = f(x) + p(Ax + g(x)),\ q \leq 1$
$f(x) = \begin{pmatrix} (x_2 - x_1) \\ (c - a - x_3)\widetilde{sgn}(x_1) \\ x_1\widetilde{sgn}(x_2) - bx_3 \end{pmatrix}$	$f(x) = \begin{pmatrix} 0 \\ c\widetilde{sgn}(x_1) - x_3\widetilde{sgn}(x_1) + cdx_2 \\ x_1\widetilde{sgn}(x_2) - bx_3 \end{pmatrix}$
$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$g(x) = \begin{pmatrix} 0 \\ -\widetilde{sgn}(x_1) \\ 0 \end{pmatrix}$

Table 2: PWL Chen systems utilized in this paper.

Even for $p := a$ (Table **??** second column), when the system does not belong to the class of systems modeled by the IVP (**??**), the PS algorithm applies to this more general class of systems:

$$D_*^q x = f(x) + p(Ax + g(x)), \quad t \in I = [0, T], \quad x(0) = x_0, \tag{9}$$

with $g(x) = (0, -\widetilde{sgn}, 0)^T$.

4 Numerical results

In this section we apply the PS algorithm to approximate stable cycles of Chen's PWL system of fractional-order (**??**) for $p := d$ the integer-order case, and $p := a$ integer and fractional-order cases. The case $p := a$ is an example of a new class of systems of integer and fractional-order where the PS algorithm still applies. For this purpose, we consider the continuously approximated Chen PWL system of fractional-order (**??**) and chose the sets $\mathcal{P}$ and $\mathcal{M}$ which generate the targeted values p^*, after which the PS algorithm applies via the scheme (**??**). The targeted values for p^* are chosen in the stable windows of bifurcation diagrams in Figure **??**. We employ an integration step-size of $h = 0.001 - 0.005$. For all phase plots, transients have been omitted. To underline numerically and computationally the match between the two attractors (A^* and A_{p^*}), overplotted phase plots, time series, histograms and Poincaré sections have been made. Also, the Hasudorff dimension, d_H [**?**] p. 114, is for all considered cases, of order of $1e - 3$, which indicates a good match between the two trajectories.

- $p := d$,

  <u>Commensurate case $q = (1, 1, 1)$</u>

  With $a = 1.15$, $c = 2$ and $b = 0.15$, the system behaves chaotically (see bifurcation diagram in Fig. **??** (a)). Let us choose $\mathcal{P}_2 = \{0.32, 0.48\}$ (Fig. **??** (a)) and $\mathcal{M} = \{1, 1\}$ with the underlying scheme, $[1p_1, 1p_2]$. The switched attractor A^* approximates the averaged attractor A_{p^*} with p^* given by (**??**): $p^* = 0.4$. The match between both attractors can be seen in the overplotted phase plots in Fig. **??** (b) and overplotted time series in Fig. **??** (e)-(g). We have a control-like scheme, since the approximated attractor A_{p^*}, is a stable cycle. The underlying attractors are a chaotic one $A_{0.32}$, and a stable one, $A_{0.48}$ (Fig. **??** (c), (d)).

- $p := a$

  Since the case $p := d$ represents the case of the class of systems modeled by (**??**), where the PS algorithm has been extensively studied in previous works, we consider next the case $p := a$, which belongs to the class of systems modeled by (**??**), and where the PS algorithm has been not applied yet. For $b = 0.15$, $c = 2$ and $d = 0.1$, as can be seen from the bifurcation diagram (Fig. **??** (b)), system dynamics are more complex than for $p := d$, presenting several direct and reverse logistic map bifurcations. In this case, due to the term $\widetilde{psgn}(x_1)$, the system is modeled by (**??**). However we will show that the PS algorithm still applies for this more general case.

  Commensurate case $q = (1, 1, 1)$

  Choosing $\mathcal{P}_2 = \{0.9, 1\}$, (Fig. **??** (a)), with $\mathcal{M} = \{1, 1\}$, the relation (**??**) gives $p^* = 0.95$, and with the scheme $[1p_1, 1p_2]$ the PS algorithm generates the switched attractor A^* which approximates the stable cycle A_{p^*}. The correspondence can be seen in the phase plots (Fig. **??** (b)) and time series (Fig. **??** (e)-(g)). This time the utilized attractors are chaotic: $A_{1.9}$ and $A_{0.9}$ (Fig. **??** (c), (d)).

  The non-uniqueness of solutions for p^* in the relation (**??**), allows different ways to obtain some desired attractor A_{p^*}. For example, the same attractor $A_{0.95}$ obtained above with the scheme $[1p_1, 1p_2]$ (with $p_1 = 0.9$ and $p_2 = 1$, values situated in closed neighborhoods of p^*), can be approximated by a switched attractor A^* obtained with p values situated relatively distant from p^*. Thus, $A_{0.95}$ can be approximated with $P_N = \{0.67, 1.51\}$ (Fig. **??** (a)) and weights $\mathcal{M} = \{2, 1\}$, i.e. the scheme $[2p_1, 1p_2]$. Phase plots (Fig. **??** (b)) and time series (Fig. **??** (e)-(g)) reveal the approximation. This time, the stable cycle A^* is obtained starting from parameter values which generate stable cycles (Fig. **??** (c), (d)).

  Also due to this same non-uniqueness, a set value p^* can be obtained with $N > 2$ elements. For example $p^* = 0.95$ can be obtained with $\mathcal{P}_6 = \{0.6, 0.7, 0.8, 1.09, 1.32, 1.4\}$ (Fig. **??** (a)) and $\mathcal{M} = \{1, 2, 2, 3, 1, 1\}$. By applying the PS algorithm with the underlying scheme $[1p_1, 2p_2, 2p_3, 3p_4, 1p_5, 1p_6]$, one obtains again a good match between the two attractors A^* and A_{p^*} (Fig. **??**).

  Incommensurate case $q = (0.99, 0.98, 0.97)$

  With $\mathcal{P}_4 = \{0.75, 1.29, 1.52, 1.6, 1.38\}$ and $\mathcal{M} = \{1, 1, 2, 2\}$, the PS scheme $[1p_1, 1p_2, 2p_3, 2p_4]$ generates the attractor A^*, which approximates A_{p^*} with $p^* = 1.38$. The match between the two attractors are revealed in Fig. **??**.

Conclusion

In this paper we introduced a new variant of Chen's system: the PWL Chen's system of fractional-order, revealing interesting new dynamical characteristics — including the coexistence of chaotic and ordered motions. To numerical cope with this system, we first must make a continuously approximation by using an algorithm based on Filippov's regularization and also by applying Cellina's Theorem.

We show that the stable attractors can be numerically approximated by starting from a set of parameter values which are periodically switched while the underlying IVP is numerically integrated.

Results are robust across a range of moderate small step sizes.

Finally, we note that the inclusion of the case $p := a$, allowed us to verify that the PS algorithm applies also to a new general class of systems modeled by (**??**).

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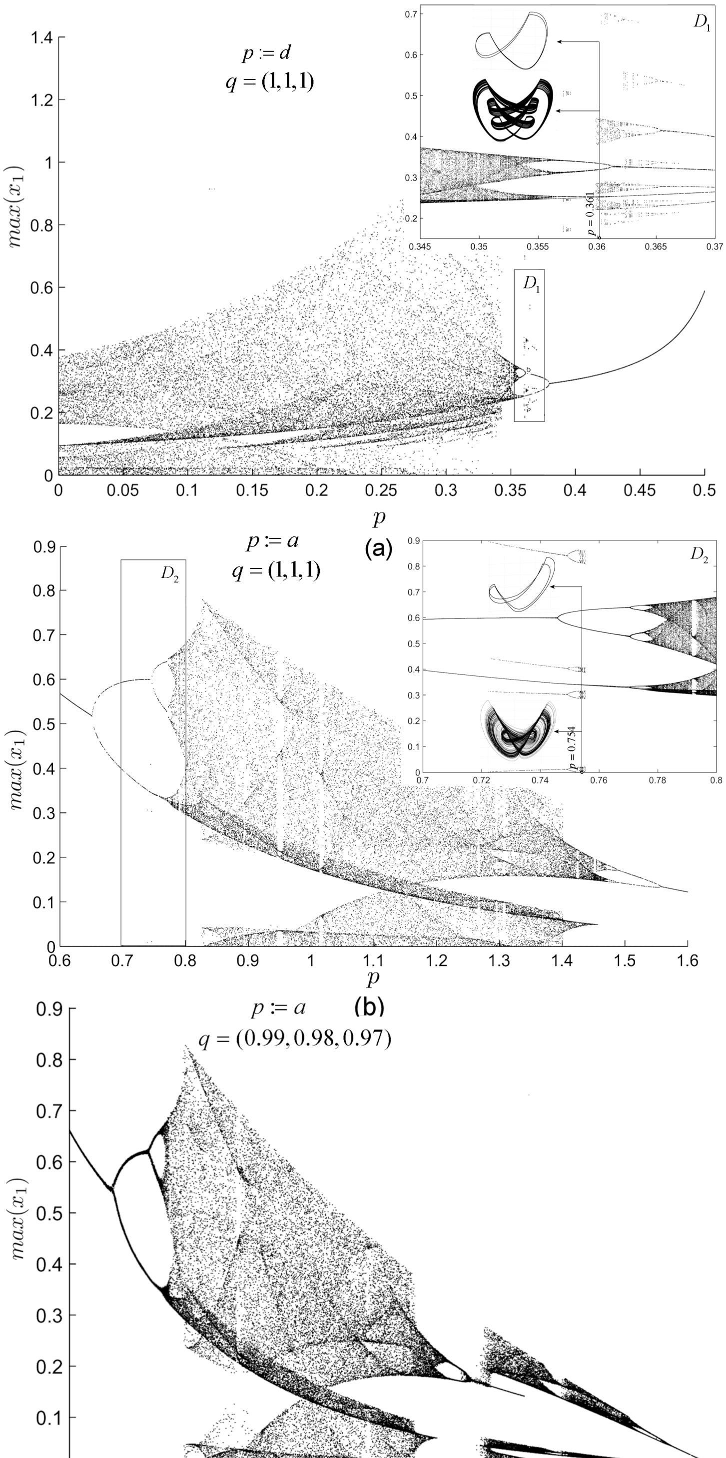

Figure 1: Bifurcation diagrams of Chen's system (**??**); (a) $p := d$, $q = (1,1,1)$ and $a = 1.15$, $c = 2$ and $b = 0.15$; (b) $p := a$, $q = (1,1,1)$ and $b = 0.15$, $c = 2$ and $d = 0.1$; (c) $p = a$, $q = (0.99, 0.98, 0.97)$ and $b = 0.15$, $c = 2$ and $d = 0.1$.

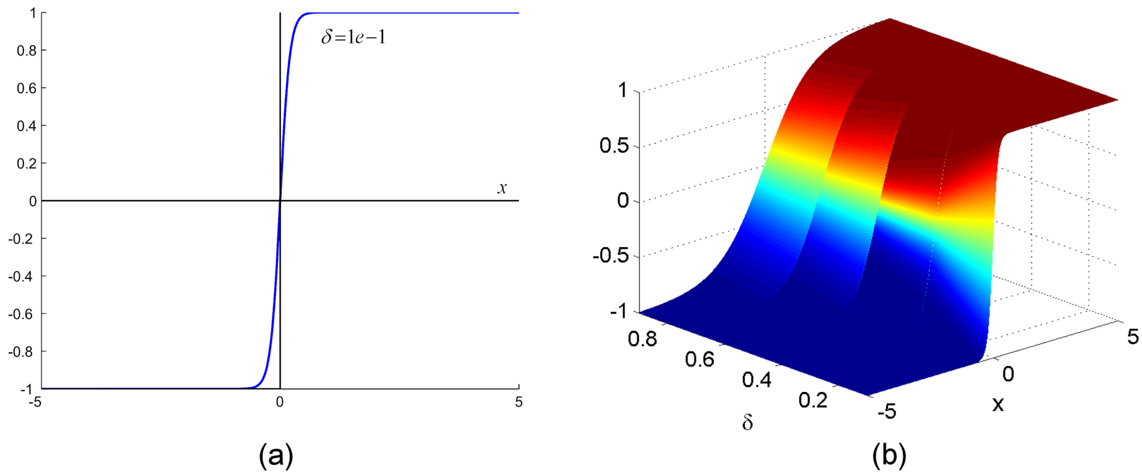

Figure 2: (a) Graph of the sigmoid function $\widetilde{sgn}$ for $\delta = 1e-1$; (b) Graph of the sigmoid function depending on δ.

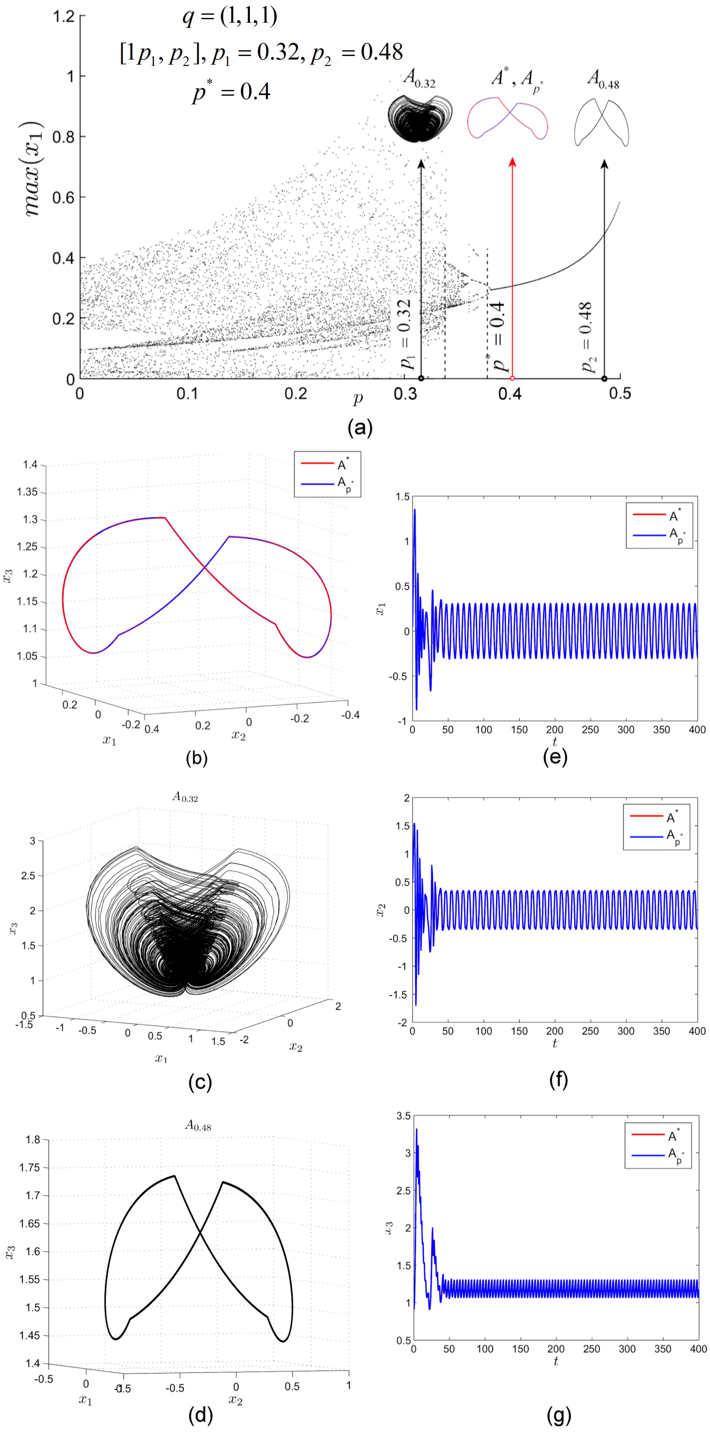

Figure 3: PWL Chen's attractor $A_{0.4}$, for $p := d$, and $q = (1, 1, 1)$, approximated with PS algorithm with schme $[1p_1, 1p_2]$ with $p_1 = 0.32$ and $p_2 = 0.48$; (a) Bifurcation diagrams with $\mathcal{P}_2$ utilized in PS algorithm; (b) Overplotted attractors A^* (red) and A_{p^*} (blue); (c) Attractor $A_{0.32}$; (d) Attractor $A_{0.48}$; (e)-(g) Overplotted time series corresponding to A^* (red) and A_{p^*} (blue).

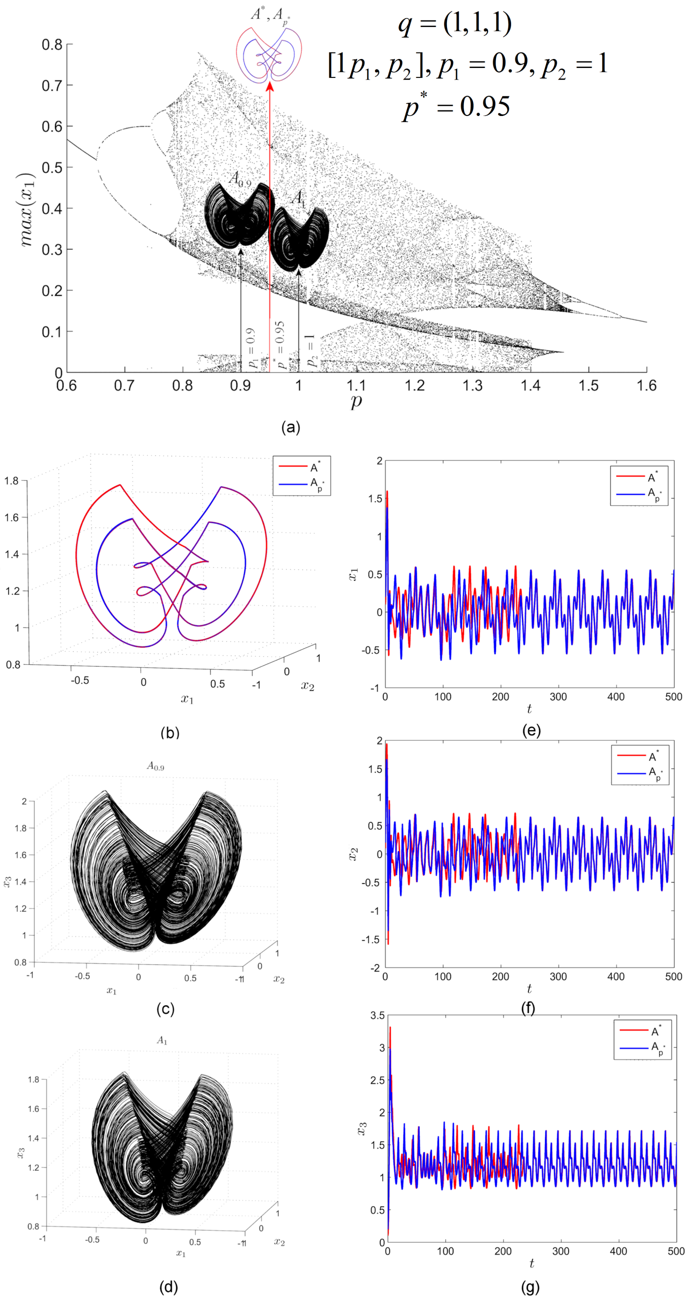

Figure 4: PWL Chen's attractor $A_{0.95}$, for $p := a$, $q = (1, 1, 1)$, approximated with PS algorithm with scheme $[1p_1, 1p_2]$ with $p_1 = 0.9$ and $p_2 = 1$; (a) Bifurcation diagram with $\mathcal{P}_2$ utilized in PS algorithm; (b) Overplotted attractors A^* (red) and A_{p^*} (blue); (c) Attractor $A_{0.9}$; (d) Attractor A_1; (e)-(g) Overplotted time series corresponding to A^* (red) and A_{p^*} (blue).

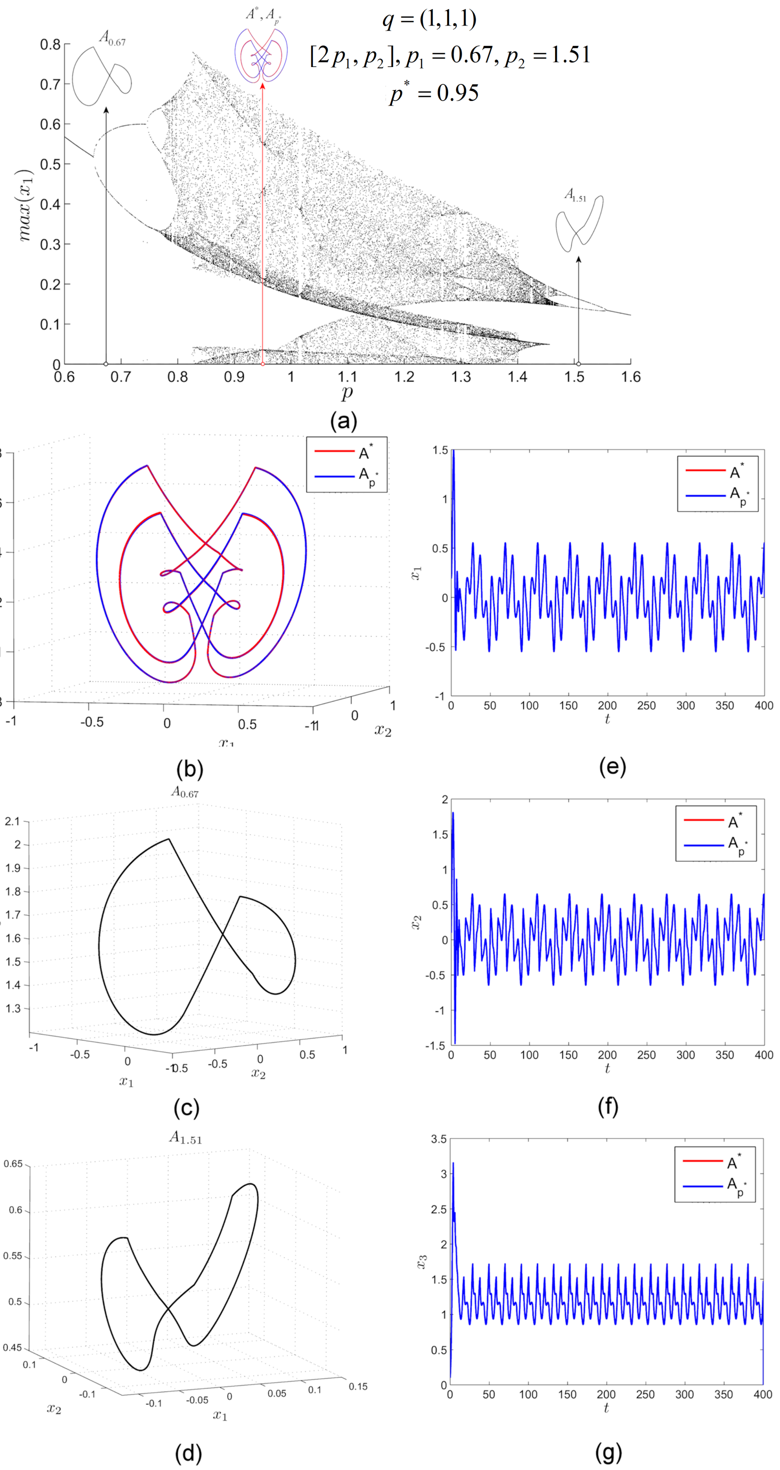

Figure 5: PWL Chen's attractor $A_{0.95}$, for $p := a$, $q = (1, 1, 1)$, approximated with PS algorithm with scehme $[2p_1, 1p_2]$ with $p_1 = 0.67$ and $p_2 = 1.51$; (a) Bifurcation diagram with $\mathcal{P}_2$ utilized in PS algorithm; (b) Overplotted attractors A^* (red) and A_{p^*} (blue); (c) Attractor $A_{0.67}$; d) Attractor $A_{1.51}$; (e)-(g) Overplotted time series corresponding to A^* (red) and A_{p^*} (blue).

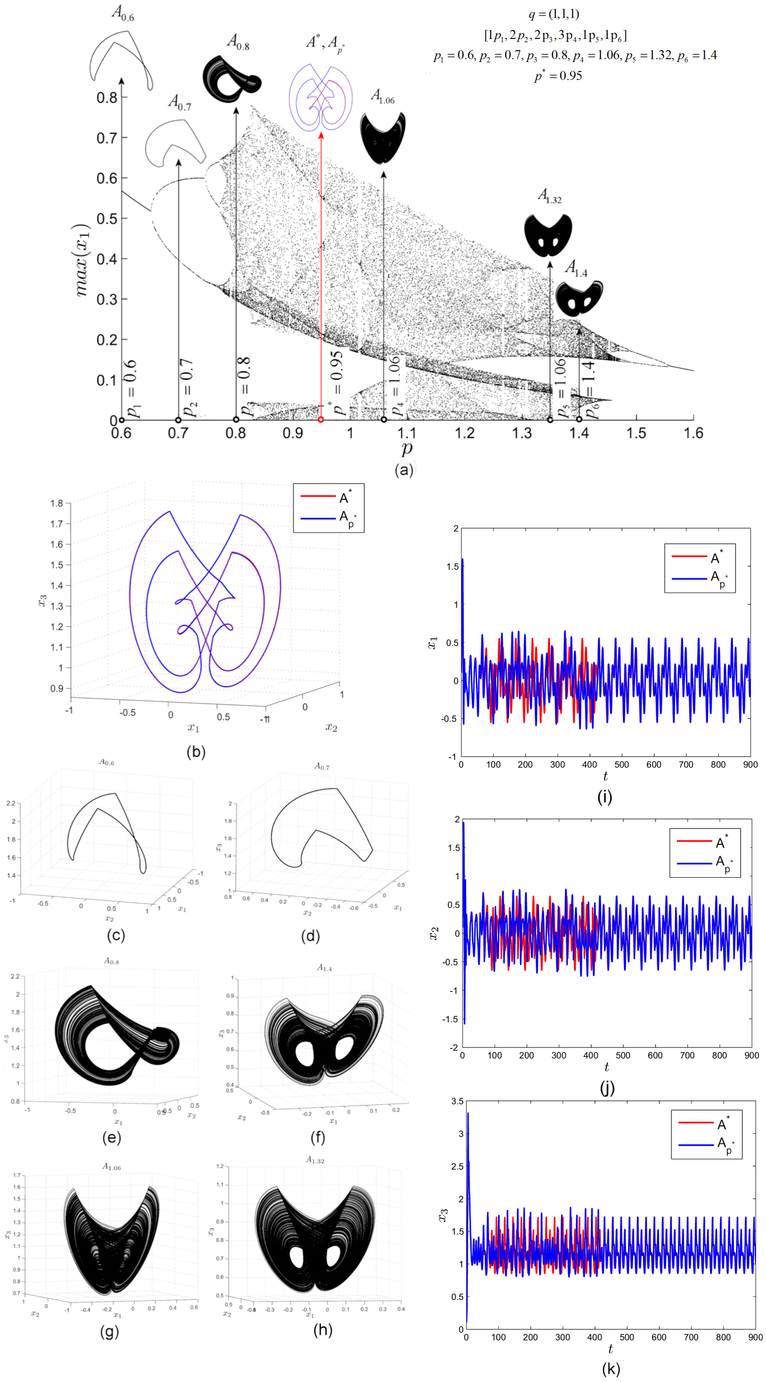

Figure 6: PWL Chen's attractor $A_{0.95}$, for $p := a$, $q = (1, 1, 1)$, approximated with PS algorithm with scheme $[1p_1, 2p_2, 2p_3, 3p_4, 1p_5, 1p_6]$ with $\mathcal{P}_6 = \{0.6, 0.7, 0.8, 1.09, 1.32, 1.4\}$; (a) Bifurcation diagram with $\mathcal{P}_2$ utilized in PS algorithm; (b) Overplotted attractors A^* (red) and A_{p^*} (blue); (c)-(h) Attractors $A_{0.6}$, $A_{0.7}$, $A_{0.8}$, $A_{1.09}$, $A_{1.32}$ and $A_{1.4}$; i-k) Overplotted time series corresponding to A^* (red) and A_{p^*} (blue). $\mathcal{M} = \{1, 2, 2, 3, 1, 1\}$.

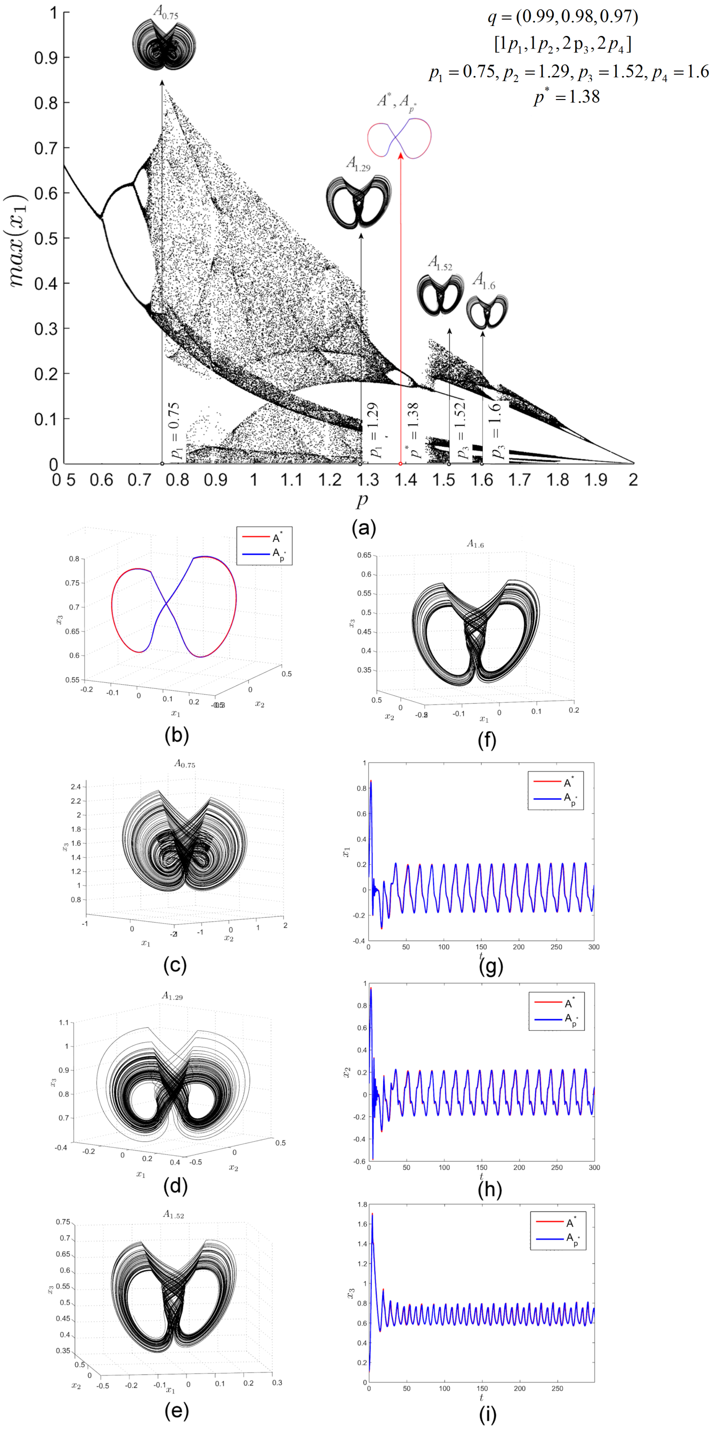

Figure 7: PWL Chen's attractor $A_{1.38}$, for $p := a$, $q = (0.99, 0.98, 0.97)$, approximated with PS algorithm with scheme $[1p_1, 1p_2, 2p_3, 2p_4]$ with $\mathcal{P}_4 = \{0.75, 1.29, 1.52, 1.6\}$; (a) Bifurcation diagram with $\mathcal{P}_2$ utilized in PS algorithm; (b) Overplotted attractors A^* (red) and A_{p^*} (blue); (c)-(g) Attractors $A_{0.75}$, $A_{1.29}$, $A_{1.52}$, $A_{1.6}$ and $A_{1.38}$; (h)-(j) Overplotted time series corresponding to A^* (red) and A_{p^*} (blue).