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Hidden chaotic attractors in fractional-order systems

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Abstract In this paper, we present a scheme for uncovering hidden chaotic attractors in nonlinear autonomous systems of fractional order. The stability of equilibria of fractional-order systems is analyzed. The underlying initial value problem is numerically integrated with the predictor-corrector Adams-Bashforth-Moulton algorithm for fractional-order differential equations. Three examples of fractional-order systems are considered: a generalized Lorenz system, the Rabinovich-Fabrikant system and a non-smooth Chua system.

keywords Hidden attractor; Self-excited attractor; Fractional-order system; Generalized Lorenz System, Rabinovich-Fabrikant system, Non-smooth Chua system

1 Introduction

The concepts of *self-excited* and *hidden* attractors have been suggested recently by Leonov and Kuznetsov (see e.g. [1–4]), which have become the subject of several works (various examples can be found in [5–12]). The basins of attraction of hidden attractors do not intersect with small neighborhoods of any equilibrium points, while a basin of attraction of a self-excited attractor is associated with an unstable equilibrium. In this context, stationary points are less important for tracking hidden attractors than for the systems with self-excited attractors. Self-excited attractors can be localized (excited) by standard computational schemes, starting from a point in a neighborhood of some unstable equilibrium. On the other hand, for localization of hidden attractors it is necessary to develop special schemes. Some known classical chaotic and regular attractors (such as Lorenz,

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Chen, Rösler, van der Pol, Sprott systems, etc.) are self-excited attractors, which can be obtained numerically with standard algorithms, and are located in some neighborhoods of unstable equilibria (their basins of attraction touch upon unstable fixed points). Hidden attractors are important in practical applications because they can lead to unexpected dynamics and instability.

Some hidden attractors can be attractors in e.g. systems with no equilibria, with only one stable equilibrium, or with coexistence of attractors in multistable engineering systems (see e.g. [9, 13–17]). Recently, coexisting hidden transient chaotic attractors have been found in the Rabinovich-Fabrikant system [18].

Uncovering all co-existing attractors and their underlying basins, when they exist, represents one of the major difficulties in locating hidden attractors. An analog of the famous 16th Hilbert problem (on the number and mutual dispositions of minimal chaotic attractors in the polynomial systems) is formulated in [19].

Hidden attractors can be regular or chaotic. In this work, we are concerned with hidden chaotic attractors. While in the above mentioned references, hidden attractors have been found for continuous-time or discrete-time systems of integer order, in this paper we present for the first time examples of hidden attractors of three-dimensional continuous-time systems of fractional order, including a generalized Lorenz system, the Rabinovich-Fabrikant (RF) system and a non-smooth Chua system.

Hidden periodic oscillations and hidden chaotic attractors have been studied e.g. in phase-locked loop [20], drilling systems [21], DC-DC convertors [22] or aircraft control system [23].

On the other hand, fractional-order systems are dynamical systems described by using fractional-order derivative and integral operators, and are studied by more and more people with growing interest. A large number of physical systems can be better modeled by means of fractional-order models [24]. Also, systems of fractional order can be found in economy [25], bioengineering [26], mechanics [27] etc. Actually, real objects or phenomena such as dielectric polarization, viscoelastic systems, percolation, polymer modeling, ultra-slow processes, electromagnetic waves, evolution of complex systems, secure communication, chaotic dynamics etc. are generally of fractional order (see e.g. [28–35]).[1]

Therefore, studying hidden chaotic attractors in systems of fractional order represents a good opportunity to deepen the new exciting and still less-explored subject of importance.

This paper is organized as follows: In Section 2, basic notions related to the stability of systems of fractional order, required to verify the attractors hiddenness characteristic and the numerical integration, are presented. Section 3 considers the hidden attractors of a generalized Lorenz system, the Rabinovich-Fabrikant system and a non-smooth Chua system. Finally a conclusion ends the paper.

[1] Note that even fractional-order dynamics allow to describe a real object more accurately than classical "integer-order" dynamics, as proved recently for the existence of stable cycles in systems of fractional order to be impossible [36, 37].

2 Stability and discretization of fractional-order systems

The considered dynamical systems are modeled by the following fractional-order initial values problem (IVP):

$$\frac{d^q}{dt^q}x(t) = f(x(t)), \quad x(0) = x_0, \quad t \in I = [0, T], \tag{1}$$

where $x : I \to \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous nonlinear function and $q \in (0, 1)$ represents the commensurate order of the derivatives. For basic knowledge on fractional calculus, one may refer to [29, 30, 38–41]. In this work, we consider the fractional derivative operator d^q/dt^q, with $q < 1$, to be *Caputo*'s derivative with starting point $t_0 = 0$ defined by [38]

$$\frac{d^q}{dt^q}x(t) = \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}x'(s)ds, \tag{2}$$

where Γ is the Euler gamma function. The use of Caputo's definition allows coupling the fractional differential equations with initial conditions in a classical form and avoids the expression of initial conditions with fractional derivatives. Note that coupling differential equations with classical initial conditions of Cauchy type not only has a clearly interpretable physical meaning but also can be measured to properly initializing simulations[2] (see [39] for more insights on this topic and relationship to the case of $q > 1$).

The right-hand side of the IVP (1) in the considered examples are Lipschitz functions, and the numerical method used in this work to integrate system (1) is the Adams-Bashforth-Moulton predictor-corrector algorithm [44]. Specifically, the algorithm works by introducing a discretization with grid points $t_i = hi, i = 0, 1, ...,$ and a preassigned step size h. For some fractional-order q, and $i = 0, 1, 2, ...,$ it first computes a preliminary approximation (predictor) denoted as x^P_{i+1} for $x(t_{i+1})$ using the formula

$$x^P_{i+1} = \sum_{j=0}^{\lceil q\rceil-1} x_0^{(j)}\frac{t^j_{i+1}}{j!} + \frac{1}{\Gamma(q)}\sum_{j=0}^{i} b_{j,i+1}f(x_j),$$

with

$$b_{j,i+1} = \frac{h^q}{q}\left((i+1-j)^q - (i-j)^q\right),$$

and then calculates the corrector value x_{i+1} by

$$x_{i+1} = \sum_{j=0}^{\lceil q\rceil-1} x_0^{(j)}\frac{t^j_{i+1}}{j!} + \frac{h^q}{\Gamma(q+2)}\left(\sum_{j=0}^{i} a_{j,i+1}f(x_j) + f(x^P_{i+1})\right),$$

where

[2] Recently, based on philosophical arguments rather than a mathematical point of view, some researchers questioned the appropriateness of using initial conditions of the classical form in the Caputo derivative [42]. However, it should be emphasized that, in practical (physical) problems, physically interpretable initial conditions are necessary and Caputo's derivative is a fully justified tool [43].

$$a_{j,i+1} = \begin{cases} i^{q+1} - (i-q)(i+1)^q, & j = 0, \\ (i-j+2)^{q+1} + (i-j)^{q+1} - 2(i-j+1)^{q+1}, & 1 \leq j \leq i, \\ 1, & j = i+1. \end{cases}$$

To define the stability of equilibria of fractional-order systems (required by the procedure to find hidden attractors), consider some equilibrium X^* and the Jacobian $J = \frac{\partial f}{\partial x}|_{x=X^*}$ evaluated at X^*. Denote by $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ the eigenspectrum and let the minimum of all arguments of the eigenvalues be $\alpha_{min} = min\{|\alpha_i|\}$, $i = 1, 2, ..., n$. Then, a stability theorem [45,46] can be stated in the following practical form:

Theorem 1 *X^* is asymptotically stable if and only if the instability measure*

$$\iota = q - 2\alpha_{min}/\pi \tag{3}$$

is strictly negative.

If $\iota \leq 0$ and the critical eigenvalues satisfying $\iota = 0$ have the geometric multiplicity one, then X^* is stable.[3]

Remark 1 If ι is positive, then X^* is unstable and the system may exhibit chaotic behavior.

3 Hidden chaotic attractors

From a computational point of view, self-excited attractors and hidden attractors are defined as follows:

Definition 1 [1–3] An attractor is called a self-excited attractor if its basin of attraction intersects with any open neighborhood of an equilibrium, otherwise it is called a hidden attractor.

Self-excited attractors can be obtained numerically with standard computational schemes, in which after transients being eliminated, the trajectories starting from neighborhoods of unstable equilibria are attracted by the attractor. In contrast, the basin of attraction for a hidden attractor is not connected with any equilibrium. Therefore, for the numerical localization of hidden attractors it is necessary to develop special analytical-numerical algorithms (see e.g. [1] and [16]). The first stage in the localization requires a harmonic linearization procedure, which allows one to modify the system such that its linear part has a periodic solution[4]. Next step is to modify the nonlinearity by introducing a small parameter. This parameter must be small enough in order to generate a periodic solution, which will be the first step of the multi-step numerical continuation procedure: construct a sequence of similar systems such that for the first (starting) system the initial

[3] The geometric multiplicity represents the dimension of the eigenspace of the corresponding eigenvalues.

[4] In many cases, one can simplify this procedure and consider instead a path in the space of parameters, such that the starting point of the path corresponds to a self-excited attractor.

point for numerical computation of oscillating solution (starting oscillation) can be obtained analytically (e.g, it is often possible to consider the starting system with self-excited starting oscillation). Then, the transformation of this starting oscillation is followed numerically in passing from one system to another and the last system will correspond to the hidden attractor.

Summarizing, to obtain a hidden attractor it is necessary to first verify that it is characterized by Definition 1: Supposing that the system admits stable and unstable equilibria (as in our considered examples). This means one should verify numerically that trajectories starting from vicinities of unstable equilibria either are attracted by stable equilibria or tend to infinity. The next step is to visualize the hidden attractor, for example by following the procedure described in [1]. However, for such as the examples considered in this work, the try-and-error method can be utilized also for plotting the hidden attractor.

In order to obtain significant (stronger) chaotic behaviors, the fractional-order q for the considered systems is chosen to be relatively high (close to 1).

3.1 Hidden chaotic attractor of a generalized Lorenz system of fractional order

For each considered system, several numerical experiments to generate sets of 100 trajectories starting from each unstable equilibrium have been done. However, for the image clarity, only representative trajectories are presented here.

The generalized Lorenz system of fractional order is a fractional variant of the generalized Lorenz system of integer order [16, 47] and is obtained from a Rabinovich system [48, 49] as follows:

$$\begin{array}{l} x_1^q = -\sigma(x_1 - x_2) - a x_2 x_3, \\ x_2^q = r x_1 - x_2 - x_1 x_3, \\ x_3^q = -x_3 + x_1 x_2, \end{array} \tag{4}$$

where $\sigma = -ar$ and $a < 0$. For $a = 0$, the system (4) coincides with the classical Lorenz system.

The system (4) can be used to describe: the convective fluid motion inside rotating ellipsoid [50], the rotation of rigid body in viscous fluid [51], the gyrostat dynamics [52], the convection of horizontal layer of fluid making harmonic oscillations, or the model of Kolmogorovs flow [53].

Due to the symmetry

$$T(x_1, x_2, x_3) \to (-x_1, -x_2, x_3), \tag{5}$$

under transformation T, each trajectory has its symmetrical (twin) trajectory with respect to the x_3-axis.

As mentioned in [16], for $r < 1$, there exists a unique equilibrium $X_0^* = (0, 0, 0)$, while for $r > 1$ there exist three equilibria: X_0^* and

$$X_{1,2}^* = (\pm x^*, \pm y^*, z^*),$$

with

$$x^* = \frac{\sigma\sqrt{\xi}}{\sigma + a\xi}, \quad y^* = \sqrt{\xi}, \quad z^* = \frac{\sigma\xi}{\sigma + a\xi},$$

where

$$\xi = \frac{\sigma}{2a^2}\left[a(r-2) - \sigma + \sqrt{(\sigma - ar)^2 + 4a\sigma}\right].$$

Let $r = 6.8$ and $a = -0.5$. Then, equilibria $X^*_{1,2}$ are given by

$$X^*_{1,2} = (\pm 3.476, \pm 1.807, 6.280),$$

and the integer-order system (4) presents a hidden attractor [16].

Here, we focus on the existence of a hidden attractor for the case of this fractional-order system. The Jacobian matrix is

$$J = \begin{bmatrix} -\sigma & \sigma - ax_3 & -ax_2 \\ r - x_3 & -1 & -x_1 \\ x_2 & x_1 & -1 \end{bmatrix}.$$

Consider the equilibrium X^*_0. The Jacobian evaluated at this point has the eigen-spectrum $\Lambda = \{2.5576, -1, -7.5576\}$ with arguments: $\alpha_1 = 0$ and $\alpha_{2,3} = \pm\pi$ and $\alpha_{min} = 0$. In this case, the instability measure (3) is $\iota = q - 2\alpha_{min}\pi/2 = q > 0$ for all $q \in (0,1)$, so the equilibrium X^*_0 is unstable.

Consider the equilibrium X^*_1 (due to the system symmetry, T, X^*_2 behaves similarly). The eigen-spectrum is $\Lambda = \{-5.9570, -0.0215 - 3.6026i, -0.0215 + 3.6026i\}$ with arguments $\alpha_1 = \pi$, $\alpha_{2,3} = \pm 1.5768$ and $\alpha_{min} = 1.5768$. Since the instability measure is $\iota = q - 2\alpha_{min}/\pi = q - 1.0038 <$ for all $q \in (0,1)$, the equilibria $X^*_{1,2}$ are asymptotically stable for all $q \in (0,1)$, so they are saddle points.

Note that the stability of equilibria $X^*_{0,1,2}$ doesn't change for any values of $q < 1$, which is similar to the integer-order case.

Consider $q = 0.995$, a value for which the system exhibits chaotic behavior. In order to verify that the generalized Lorenz system (4) has a hidden attractor, we have to verify that there exist small neighborhoods of the unstable equilibrium X^*_0, orbit from which are all attracted by the stable equilibria $X^*_{1,2}$ (Fig. 1). As can be seen in Fig. 1 (a), trajectories exiting from a small vicinity of X^*_0 either tend to X^*_1 (red plot), or to X^*_2 (blue plot). In the detail in Fig. 1 (b), the considered 50 trajectories starting from a vicinity of ray $\delta = 0.3$ centered at X^*_0 show how they are attracted either by X^*_1, or X^*_2. The hidden chaotic attractor H is colored in green.

3.2 Hidden chaotic attractor of the Rabinovich-Fabrikant system

The fractional-order RF system [54, 55] is modeled by

$$\begin{aligned} x_1^q &= x_2\left(x_3 - 1 + x_1^2\right) + ax_1, \\ x_2^q &= x_1\left(3x_3 + 1 - x_1^2\right) + ax_2, \\ x_3^q &= -2x_3\left(b + x_1x_2\right), \end{aligned} \tag{6}$$

with $a > 0$ and b being the bifurcation parameter. The system, initially designed as a physical system, describes the stochasticity arising from the modulation instability in a dissipative medium. However, as revealed numerically in [54] and [55], the system of integer order presents unusual and extremely rich dynamics, including multistability, an important ingredient for potential existence of hidden attractor.

The equilibria are X_0^* and

$$X_{1,2}^* \left(\mp x_{1,2}^*, \pm y_{1,2}^*, z_{1,2}^*\right), \quad X_{3,4}^* \left(\mp x_{3,4}^*, \pm y_{3,4}^*, z_{3,4}^*\right), \tag{7}$$

where

$$x_{1,2}^* = \sqrt{\frac{bR_1 + 2b}{4b - 3a}}, \quad y_{1,2}^* = \sqrt{b\frac{4b - 3a}{R_1 + 2}}, \quad z_{1,2}^* = \frac{aR_1 + R_2}{(4b - 3a)\,R_1 + 8b - 6a},$$

and

$$x_{3,4}^* = \sqrt{\frac{bR_1 - 2b}{3a - 4b}}, \quad y_{3,4}^* = \sqrt{b\frac{4b - 3a}{2 - R_1}}, \quad z_{3,4}^* = \frac{aR_1 - R_2}{(4b - 3a)\,R_1 - 8b + 6a},$$

with $R_1 = \sqrt{3a^2 - 4ab + 4}$ and $R_2 = 4ab^2 - 7a^2b + 3a^3 + 2a$.

Let $a = 0.1$ and $b = 0.2876$. Then, the equilibria $X_{1,2,3,4}^*$ are

$$X_{1,2}^* = (\mp 1.1600, \pm 0.2479, 0.1223), \quad X_{3,4}^* = (\mp 0.0850, \pm 3.3827, 0.9953).$$

It is easy to see that the system exhibits the same symmetry T defined in (5), as for the case of the generalized Lorenz system (4). Therefore, we can consider the stability of X_1^* and X_3^* only.

As required by Definition 1, we have to determine the stability of all equilibria. The Jacobian is

$$J = \begin{bmatrix} 2x_1x_2 + a & x_1^2 + x_3 - 1 & x_2 \\ -3x_1^2 + 3x_3 + 1 & a & 3x_1 \\ -2x_2x_3 & -2x_1x_3 & -2(x_1x_2 + b) \end{bmatrix}.$$

Consider, first, the equilibrium X_0^*. The spectrum of eigenvalues is $\Lambda = \{-0.5752, 0.1-i, 0.1+i\}$, with arguments: $\alpha_1 = \pi$, $\alpha_{2,3} = \pm 1.4711$, $\alpha_{min} = 1.4711$. The instability measure is $\iota = q - 2\alpha_{min}/\pi = q - 0.9365 > 0$ for $q > 0.9365$. Therefore, X_0^* is an unstable focus-saddle only for $q > 0.9365$, while for smaller values of q the equilibrium X_0^* is stable.

The eigen-spectrum of the equilibrium X_1^* is $\Lambda = \{-0.2562, -0.0595-1.4731i, -0.0595+1.4731i\}$ with arguments: $\alpha_1 = \pi$, $\alpha_{2,3} = \pm 1.6112$, $\alpha_{min} = 1.6112$ and $\iota = q - 1.0257 < 0$ for all $q \in (0,1)$. Therefore, $X_{1,2}^*$ are stable for all $q \in (0,1)$.

Finally, the spectrum of the equilibrium X_3^* is $\Lambda = \{0.1981, -0.2866-4.7743i, -0.2866+4.7743i\}$ with arguments: $\alpha_1 = 0$, $\alpha_{2,3} = \pm 1.6308$ and $\alpha_{min} = 0$. In this case, $\iota = q > 0$ and, therefore, $X_{3,4}^*$ are unstable for all $q \in (0,1)$.

Taking account on the instability of the equilibrium X_0^*, we consider $q = 0.998$.

For the considered parameter values and fractional order, beside the hidden attractor H (green plot in Fig. 2 (a)) the system presents unusual behavior such as “unbounded self-excited attractors” $Y_{1,2}^*$ (Fig. 2 (a)) which, due to the resemblance with saddles, are called “virtual” repelling saddles [55]. As shown in [55], these saddles-like attractors exist for relatively large domains of parameters a and b.

To check that the chaotic attractor H is hidden (see the detailed image in Fig.2 (b), where the equilibria $X_{0,1,2,3,4}^*$ beside H can be viewed), we have to verify numerically that all trajectories starting from small vicinities of all unstable equilibria (X_0^* and $X_{3,4}$) either diverge to infinity or are attracted by the stable equilibria $X_{1,2}^*$. As can be seen in Fig. 2 (b) and Fig. 2 (c), all trajectories starting from the vicinity of X_0^* either diverge to infinity (grey plot), or converge to the stable equilibria $X_{1,2}$ (dotted blue and dotted red plot, respectively). From Fig. 2 (b) and Fig. 2 (c), one can see that the trajectories starting from a vicinity of X_3^* (X_4^* leads to similar behavior) tend either to the "virtual" repelling saddle Y^*, or are attracted by $X_{1,2}^*$ (blue plot and red plot respectively). The sizes of the vicinities in this case have been chosen as $\delta = 0.1$, and for the sake of image clarity, only representative trajectories are plotted.

3.3 Hidden chaotic attractor of a non-smooth Chua system of fractional order

Consider the non-smooth Chua system modeled by [56, 57]

$$\begin{aligned} x_1^q &= \alpha(x_2 - x_1 - m_1x_1 - \psi(x_1)),\\ x_2^q &= x_1 - x_2 + x_3,\\ x_3^q &= -(\beta x_2 + \gamma x_3), \end{aligned} \tag{8}$$

with

$$\psi(x_1) = (m_0 - m_1)sat(x_1),$$

and

$$\begin{aligned} m_0 &= -0.1768,\\ m_1 &= -1.1468,\\ \alpha &= 8.4562,\\ \beta &= 12.0732,\\ \gamma &= 0.0052. \end{aligned}$$

After almost 30 years of its first investigation of Chua's circuits of integer or fractional order, only self-excited attractors have been found. However, later it was shown (see e.g. [56]) that Chua's circuits of integer order has hidden chaotic attractors, with a positive largest Lyapunov exponent.

The system is continuous non-smooth because of the function ψ. But, its right-hand side is locally Lipschitz from the absolute value operator in $\psi(x_1)$, so that the ABM method can be applied [39].

Equilibria are: $X_0^* = (0, 0, 0)$ for $|x| < 1$ and $X_{1,2}^* = (\pm 6.5883, \pm 0.0029 \mp 6.5855)$ for $|x| > 1$, and the Jacobian is

$$J = \begin{bmatrix} -\alpha - \alpha m_1 & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{bmatrix}, \quad \text{for} \quad |x| > 1,$$

and

$$J = \begin{bmatrix} -\alpha - \alpha m_1 - m_0 + m_1 & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{bmatrix}, \quad \text{for} \quad |x| < 1.$$

The eigen-spectrum of X_0^* is $\Lambda = \{-7,9587, -0.0038 \pm 3.2494i\}$ and $\iota = q - 1.0008 < 0$ for all $q \in (0,1)$. Therefore, X_0^* is asymptotically stable.

For $X_{1,2}^*$, $\Lambda = \{2.2193, -0.9916 \pm 2.4068i\}$ and $\iota = q > 0$ for all $q \in (0,1)$ and the equilibria are unstable.

For $q = 0.9998$, the numerical results are presented in Fig. 3 (a), while the detail in Fig. 3 (b) reveals the tendency of all trajectories starting from a small vicinity of unstable equilibria (of size $\delta = 0.01$) tend either to the stable equilibrium X_0^* (red plot) or diverge to infinity (blue plot).

4 Conclusion

Our new results show that smooth or non-smooth fractional-order three-dimensional systems of commensurates order can exhibit hidden chaotic attractors. The presented approach can be successfully implemented also for other systems of fractional order. Similarly, one can search hidden attractors for fractional-order systems of incommensurate orders. In addition, we have revealed that the unusual and extremely rich dynamics ("virtual" saddles-like) of the RF system, found previously for the integer-order system, persist for the corresponding fractional-order variant.

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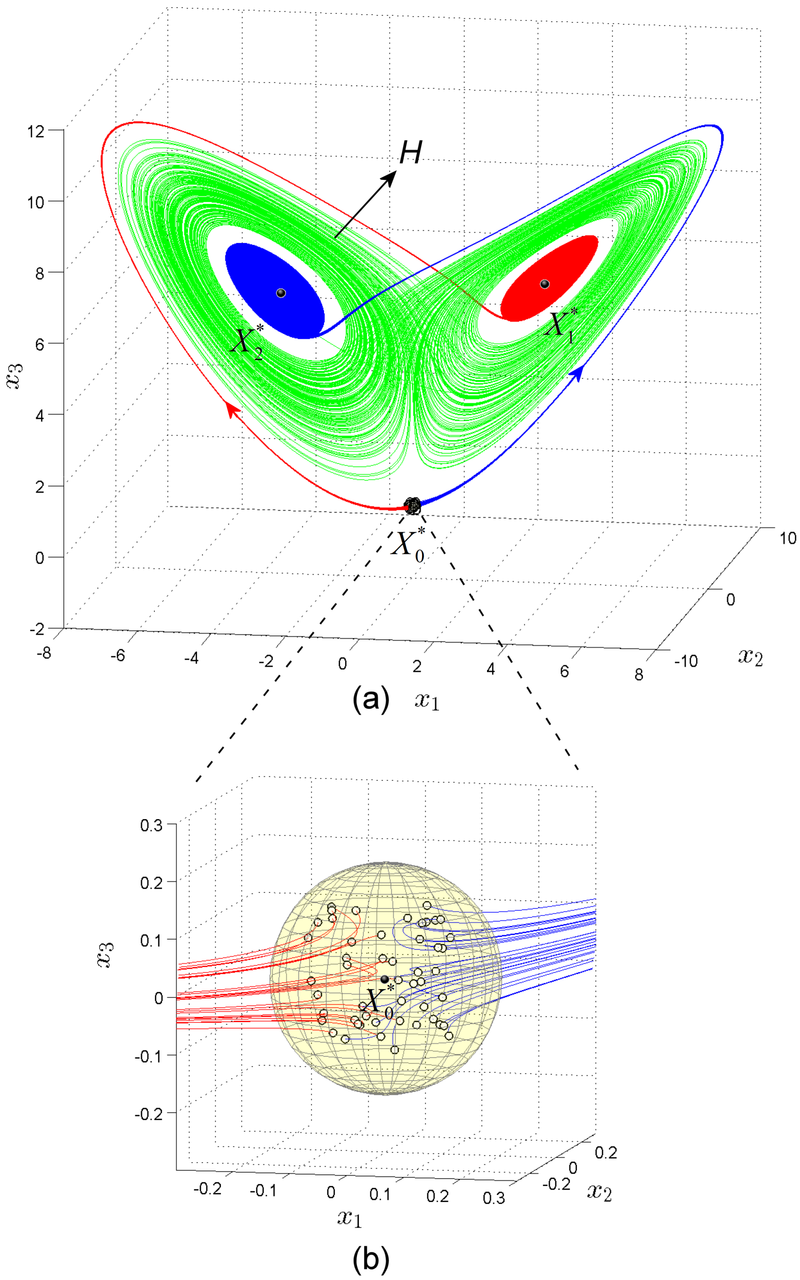

Fig. 1 (a) Hidden attractor of the generalized Lorenz system of fractional-order (green). The red and blue trajectories, starting from a small vicinity of the unstable equilibrium X_0^*, are attracted by the stable equilibria $X_{1,2}^*$. (b) The detailed image shows how the considered 50 trajectories tend either to X_1^* (red), or to X_2^* (blue).

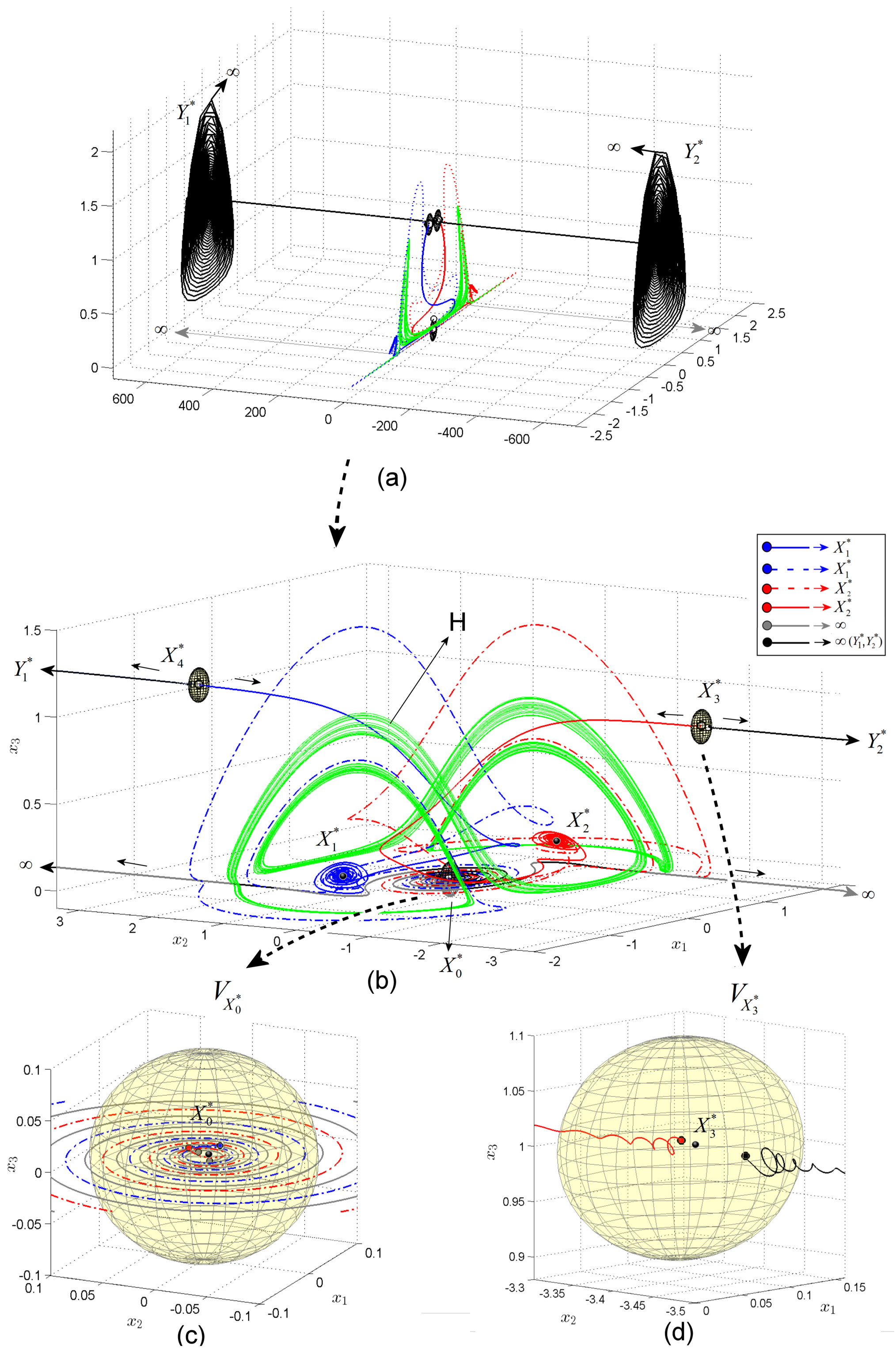

Fig. 2 (a) Hidden attractor H (green) and "virtual" repelling saddles $Y_{1,2}^*$ (black) of the generalized RF system of fractional-order. (b) The detailed image presents the hidden attractor, equilibria, and the trajectories starting from vicinities of unstable equilibria $X_{0,3,4}^*$. (c) Detailed region of the unstable equilibrium X_0^* revealing four trajectories, which tend either to infinity (grey) or to equilibria $X_{1,2}^*$ (dotted blue and dotted red respectively). (d) Detailed region of the unstable equilibrium X_3^* revealing two trajectories, which tend either to infinity (black) or to equilibria $X_{1,2}^*$ (blue and red respectively).

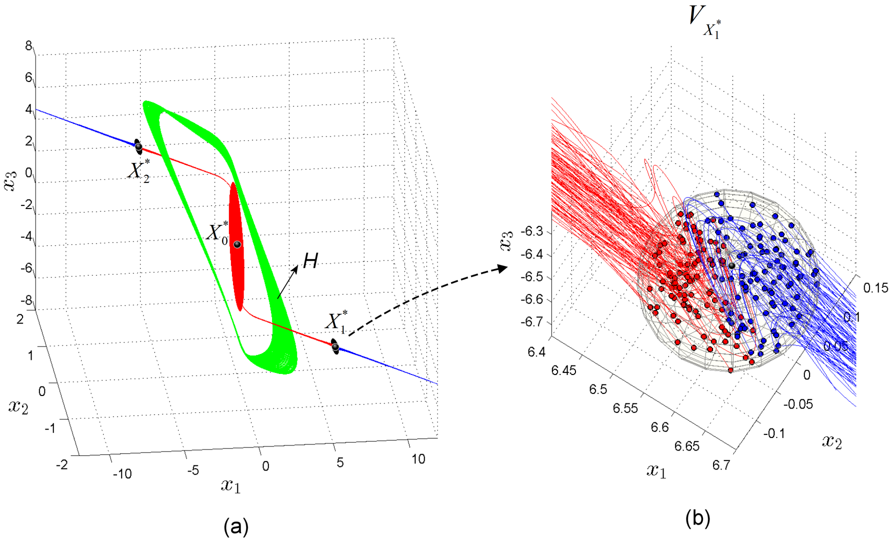

Fig. 3 Hidden attractor (green) of the non-smooth Chua system. (b) The detailed image presents 200 trajectories starting from a vicinity of the equilibrium X_1^* tending either to X_0^* (red) or to infinity (blue).