Sustaining stable dynamics of a fractional-order chaotic financial system by parameter switching

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Abstract

In this paper, a simple parameter switching (PS) methodology is proposed for sustaining the stable dynamics of a fractional-order chaotic financial system. This is achieved by switching a controllable parameter of the system, within a chosen set of values and for relatively short periods of time. The effectiveness of the method is confirmed from a computer-aided approach, and its applications to chaos control and anticontrol are demonstrated. In order to obtain a numerical solution of the fractional-order financial system, a variant of the Grünwald-Letnikov scheme is used. Extensive simulation results show that the resulting chaotic attractor well represents a numerical approximation of the underlying chaotic attractor, which is obtained by applying the average of the switched values. Moreover, it is illustrated that this approach is also applicable to the integer-order financial system.

Keywords: Parameter switching, Financial system, Chaos control, Chaos

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anticontrol, Fractional-order system, Grünwald-Letnikov scheme

1. Introduction

Today, many economists still focus on linear dynamics (e.g., using the Hartman–Grobman theorem), thinking of that nonlinear dynamics are intractable although the economic world is by nature nonlinear. Nevertheless, the intrinsic relation between chaos theory and finance theory has been widely explored since the pioneering work of Smale in 1953 [1]. As a result, financial systems are commonly modeled by continuous-time chaotic systems such as the forced Van-der-Pol model [2], Behrens-Feichtinger model [3], Cournot-Puu model [4], IS-ML model [5], and so on (see also [6–9] and references therein). In addition, many recent studies on economics have demonstrated the adverse effect of chaotic dynamics on economic systems.

Due to the instability of a periodic solution, bifurcation, or other typical phenomena which could appear in chaotic economic systems, some measures and actions are required. Many researchers suggested applying chaos control in financial systems in order to improve their performances such as preserving stability. Indeed, controlling a chaotic market model may lead to economic efficiency. Therefore, interest in suppressing chaos in economic models has raised from the scientific community [10–14].

In this paper, the study is devoted to the chaotic financial system introduced in [16], which is originally of integer-order, but lately being extended to fractional-order in [17]. The system is described by the following differential equations:

$$
\begin{aligned}
\frac{dx_1^{q_1}}{dt^{q_1}} &= x_3 + (x_2 - p)x_1, \\
\frac{dx_2^{q_2}}{dt^{q_2}} &= 1 - bx_2 - x_1^2, \\
\frac{dx_3^{q_3}}{dt^{q_3}} &= -x_1 - cx_3,
\end{aligned}
\tag{1}
$$

where p, b and c are nonnegative coefficients with physical meanings and significance clearly explained in [15]; $q = (q_1, q_2, q_3)^T$ represents the fractional order of the derivatives, in which $q_i \in (0, 1]$, with $q_i = 1$, $i = 1, 2, 3$, representing the integer-order case.

For the integer-order financial system (1), i.e., with $q_i = 1$, $i = 1, 2, 3$, its local topological structure and bifurcation have been studied in detail (see [15, 16]). For its fractional-order version, the nonlinear dynamics have also been analyzed in [17, 18]. Furthermore, this financial system model has been investigated regarding chaos control and synchronization in [14, 19].

In this paper, we show numerically that any stable attractor of the financial system (1) can be approximated by switching p within a set of chosen values in deterministic and relatively small time intervals. Compared to other methods, such as OGY-like schemes, where unstable periodic orbits are "forced" to become stable, here one obtains whatever stable attractor that is desirable.

The system (1) can be reformulated as the following general initial value problem (IVP):

$$\frac{d^q}{dt^q}x(t) = f(x(t)) = g(x(t)) + pAx(t), \quad x(0) = x_0, \quad t \in I = [0, T], \tag{2}$$

where $x : I \to \mathbb{R}^3$, $g : \mathbb{R}^3 \to \mathbb{R}^3$ is a continuous nonlinear function, A is a 3×3 real constant matrix, and p is a tunable real parameter to be used for control by switching its values later.

The IVP (2) is useful for describing a large class of well-known dynamical systems of integer-order or fractional-order, for example the Lorenz, Rössler, Chen, Chua systems, to name just a few.

Referring to (1), one has

$$g(x) = \begin{pmatrix} x_3 + x_1 x_2 \\ 1 - bx_2 - x_1^2 \\ -x_1 - cx_3 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

When $q = 1$, system (2) corresponds to a classical first-order IVP, which can be numerically solved by standard methods, such as Runge-Kutta. On the other hand, for $q \in (0, 1)$, system (2) becomes an IVP of fractional-order, presented as fractional differential equation (FDE). In this case, we consider the fractional derivative operator d^q/dt^q as being the Caputo's derivative with starting point $t_0 = 0$, namely,

$$\frac{d^q}{dt^q} x(t) = \frac{1}{\Gamma(1-q)} \int_0^t \big(t - s\big)^{-q} x'(s) ds, \tag{3}$$

where Γ is the Euler gamma function (for basic knowledge on fractional calculus, one may refer to [20–25]). The use of the Caputo's approach allows coupling the FDE with initial conditions in a classical form as in (2) and, unlike the Riemann–Liouville (RL) definition

$$^{RL}D_0^q x(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t \big(t - s\big)^{-q} x(s) ds,$$

it avoids the expression of initial conditions with fractional derivatives. However, under the assumption that x is absolutely continuous, the Caputo and RL definitions are related by a relationship involving the initial condition

$$\frac{d^q}{dt^q} x(t) = {}^{RL}D_0^q \big(x(t) - x(0)\big) \tag{4}$$

(one can refer to [20] for more insights on this topic and for the extension of the above definitions and relationship to the case $q > 1$). Another approach to introduce derivatives of non–integer order is the Grünwald–Letnikov (GL) operator

$$ {}^{GL}D_0^q x(t) = \lim_{N\to\infty} h_N^{-q} \sum_{k=0}^{N} \omega_k^{(q)} x(t - kh_N), \quad h_N = t/N, \tag{5} $$

where $\omega_k^{(q)}$ are the coefficients in the power series expansion of $(1-\xi)^q = \sum_{k=0}^{\infty} \omega_k^{(q)} \xi^k$ and they can be calculated as follows:

$$ \omega_k^{(q)} = (-1)^k \binom{q}{k}, \quad \text{with } \binom{q}{k} = \frac{\Gamma(q+1)}{\Gamma(k+1)\Gamma(q-k+1)}, \quad k = 0, 1, 2, \ldots . $$

The most attractive feature of the GL definition is that it can be used for practical computation by simply truncating, the summation in (5) to a finite N. Since under minimal assumptions of continuity RL and GL definitions coincides [20], the GL discretization scheme can be used, by exploiting (4), to provide numerical solutions of Caputo–type FDEs (2) in the form of

$$ \sum_{k=0}^{m} \omega_k^{(q)} \big(x_{m-k} - x_0\big) = h^q f(x_m) $$

where x_m stands for the numerical approximation of $x(mh)$. An explicit counterpart of the above implicit GL scheme can be obtained by simply replacing $f(x_m)$ with $f(x_{m-1})$. These are convergent schemes with order 1 for any $q > 0$ [26, 27].

As will be demonstrated via numerical simulations later, by switching the parameter p in (2) within some chosen finite set of values, the generated attractor can well approximate the underlying attractor. Thus, obtaining and sustaining stable dynamics in a system modeled by (2) can be realized by choosing some stable cycles of the system corresponding to certain values of p, denoted as p^*. The parameter switching (PS) algorithm can then be designed and applied by obtaining an adequate set of switching values p and their underlying switching time subintervals, such that their average value becomes p^*.

Using the PS algorithm can generate many possible dynamics, including chaotic attractors, from a given system modeled by (2). This attractor synthesis is possible due to the convergence property of the numerical solution, subjected to the switching of p, to the corresponding solution with the average value of p (i.e., p^*). A proof of the convergence, based on the averaging method, can be found in [28].

The PS algorithm has recently been extended to systems of integer-order and fractional-order, either continuous or discontinuous, with respect to the state variables. The relevant results have been verified computationally with phase plots, time series, Poincaré sections, histograms, as well as the Hausdorff distance [29].

It is worth mentioning that there are many potential applications for the PS algorithm, for example, to obtain some particular attractors when the desired value p^* is not obtainable from the model; to explain the complexity of some natural phenomena with intrinsic switching modes.

The rest of this paper is organized as follows. In Section 2, the PS algorithm and its convergence are revisited. Numerical implementation of the algorithm for both integer-order and fractional-order systems is then presented in Section 3. In Section 4, the PS is applied as a chaos control technique to the financial system (1) so as to obtain stable dynamics. Finally, some conclusions are drawn in the last section.

2. The Parameter Switching Algorithm

First, the procedure of the PS algorithm [29] is briefly reviewed.

Assuming that the IVP (2) is solved by numerically integrating the system on interval I for the given initial x_0, the PS algorithm is applied by switching p periodically within a set of chosen real values $\mathcal{P}_N = \{p_1, p_2, ..., p_N\}$ with $N > 1$.

Convergence of the PS algorithm

For illustration and completeness, the analytical proof of the convergence of the PS algorithm is sketched below, for the integer-order case, which was presented in [28].

Consider the IVP (2) with $q = 1$ and assume that the standard conditions (e.g. Lipschitz continuity) for existence and uniqueness of its solution hold. Then, one has

$$\dot{x}(t) = f(x(t)) = g(x(t)) + pAx(t), \quad x(0) = x_0, \quad t \in I. \tag{6}$$

where $x \in \mathbb{R}^3$.

When p is varied periodically based on the PS algorithm, equation (6) is converted as follows:

$$\dot{y}(t) = g(y(t)) + p(t/\lambda)Ay(t), \quad y(0) = y_0, \quad t \in I, \tag{7}$$

where $y \in \mathbb{R}^3$ and λ is a small positive real number.

Parameter p can be considered as a piecewise constant periodic function with period T_0 and its average value p^* can be computed by

$$p^* = \frac{1}{T_0} \int_t^{t+T_0} p(u)du, \quad t \in I.$$

Consider the *average* model of (6) with p being replaced with p^*, namely,

$$\dot{x}(t) = g(x(t)) + p^* A x(t), \quad x(0) = x_0, \quad t \in I. \tag{8}$$

Based on the averaging theory [30], it is proved in [28] that the solutions of (7) and (8), starting with same initial conditions $x_0 = y_0$, differ by less than $\alpha\lambda^2$ for some constant α when λ is sufficiently small.

Numerical Implementation of the PS Algorithm

The simplest way to implement the PS algorithm is to partition interval I into subintervals of lengths being multiples of h, where $h > 0$ is the step size of the numerical method used.

An example is depicted in Figure 1. It is to set $p = p_1$ in the first time subinterval of length $m_1 h$, $p = p_2$ in the next subinterval of length $m_2 h$, and so on, until $p = p_N$ in the N-th time interval of length $m_N h$ (For the example in Figure 1, $N = 3$). The "weights" m_i, $i = 1, 2, ..., N$, are some positive integers. These N subintervals repeat until the time interval I is completely covered.

Symbolically, for a fixed h, the switching scheme can be represented by

$$[m_1 p_1,\ m_2 p_2,\ ...,\ m_N p_N]. \tag{9}$$

For example, $[2p_1,\ 1p_2]$ means that, the underlying IVP is integrated with p being

$$p_1,\ p_1,\ p_2,\ p_1,\ p_1,\ p_2,\ ...$$

for each integration step time h.

Notation 1. *[29] Let p^* be the average of the switched values under scheme (9), which is computed by*

$$p^* = \frac{\sum_{i=1}^{N} m_i p_i}{\sum_{i=1}^{N} m_i}. \tag{10}$$

The obtained solution under the switching scheme (9) approximates the solution generated with $p = p^*$.

To implement numerically the PS algorithm for the integer-order financial systems modeled by the IVP (2), some standard numerical methods such as Runge-Kutta with fixed step size can be used. In this paper, the 4th-order Runge-Kutta method is used[1].

On the other hand, to solve the fractional-order setting, a modified version of the Grünwald-Letnikov (GL) discretization scheme is suggested to apply. This scheme has already been employed in [31] for numerical simulations on the fractional-order Chua systems.

Referring to the financial system (1), the following equations can be obtained:

$$\begin{cases} x_{m,1} = x_{0,1} - \sum_{k=1}^{m} \omega_k^{(q)} \big(x_{m-k,1} - x_{0,1}\big) + h^q \big(x_{m-1,3} + (x_{m-1,2} - p)x_{m-1,1}\big), \\ x_{m,2} = x_{0,2} - \sum_{k=1}^{m} \omega_k^{(q)} \big(x_{m-k,2} - x_{0,2}\big) + h^q \big(1 - b x_{m-1,2} - x_{m,1}^2\big), \\ x_{m,3} = x_{0,3} - \sum_{k=1}^{m} \omega_k^{(q)} \big(x_{m-k,3} - x_{0,3}\big) + h^q \big(-x_{m,1} - c x_{m-1,3}\big), \end{cases} \tag{11}$$

where $x_{m,i}$ denotes the i-th component of x_m, approximating $x(mh)$.

As shown in (11), the problem can be solved by evaluating each component of $x_{m,i}$ one by one sequentially, making the scheme more effective. Although its convergence rate is of order one, this is a quite typical method for solving the fractional–order differential equations. It is also worth noting that some schemes, such as the Adams-Bashforth-Moulton scheme implemented in the Matlab code FDE12 [32], have convergence order of $(1 + q)$.

[1] Adaptive step-size methods can also be used to implement the PS algorithm.

However, their procedures are much more complicated and computationally expensive and the stability properties are not well established [33].

3. Obtaining and Sustaining Stable and Chaotic Dynamics

Due to the assumed uniqueness condition, to each initial condition x_0 belonging to some attraction basin, a unique solution of (2) can be obtained by the Runge–Kutta or the modified GL scheme, depending on whether (2) is of integer-order or fractional-order. After neglecting a sufficiently long transient period of time, the solution approximates a unique attractor on the considered basin of attraction.

Notation 2. *[29] An* attractor*, as usual for computational purposes, is understood as the numerical approximation of the solution starting from some initial condition x_0, which characterizes asymptotic dynamics of the systems with neglected transients.*

Now, consider $\mathcal{P}_N = \{p_1, p_2, ..., p_N\}$. Attractors of the set $\mathcal{A}_N = \{A_{p_1}, A_{p_2}, ..., Ap_N\}$ can be obtained, uniquely corresponding to $\mathcal{P}_N$. It is natural to consider a monotonic bijection between $\mathcal{P}_N$ and $\mathcal{A}_N$, which induces the relation order of $\mathcal{P}_N$ into $\mathcal{A}_N$ [2]. Therefore, denoting p_{min} and p_{max} as the minimum and maximum values of $\mathcal{P}_N$, respectively, the real interval (p_{min}, p_{max}) corresponds to the attractor interval $(A_{p_{min}}, A_{p_{max}})$.

Notation 3. *[29] Let A^o denote the* synthesized attractor *obtained with a switching scheme using the PS algorithm, and A^* be the* average attractor *obtained when p is replaced with p^*.*

[2] The analytical proof of bijection existence represents the subject of a present work.

Based on the PS algorithm convergence [28] and being verified by overplotted phase plots, time series, Poincaré section and the the Hausdorff distance [29], one can obtain the following property.

Property 4. *[29] For a given set P_N and weights $m_1, m_2, ..., m_N$, with $N > 1$, the synthesized attractor A^o well approximates the average attractor A^*.*

Remark 5. *i) The value of p^* given by (10) is a convex combination of p_i, $i = 1, 2, ..., N$: $p^* = \sum_{i=1}^{N} \alpha_i p_i$, where $\alpha_i = \frac{m_i}{\sum_{i=1}^{N} m_i}$, and $\sum_{i=1}^{N} \alpha_i = 1$. Therefore, given $\mathcal{P}_N$ and weights $m_1, m_2, ..., m_N$, p^* belongs inside the real interval (p_{min}, p_{max}). Moreover, due to the mentioned order and bijection, the same happens for A^o, which falls inside the interval $(A_{p_{min}}, A_{p_{max}})$. This property essentially indicates a kind of strong robustness of the PS algorithm to parameter switchings.*

ii) For the convexity property of p^*, the PS method can be applied with scheme (9) not only periodically but also randomly. For example, assuming that the designed weight of p_i is given by m_i for a targeted average value of p^*, the usage of p_i in the model can be chosen with a probability of m_i. Then, with a sufficiently long period of time, one has

$$p'^* = \frac{\sum m_i{}' p_i}{\sum m_i{}'}, \tag{12}$$

where m_i' represent the total number of occurrence for each p_i and $p'^* \approx p^*$.

As stated in Property 4, the PS algorithm allows approximating every attractor of any system modeled by (2). Therefore, the PS algorithm can be used accordingly as a "chaos control" or "anti-control" technique. Hereafter, these notions will be discussed.

The only condition for chaos control (anti-control) is that, between the chaotic (periodic) windows wherefrom the switched values are chosen, there exists at least one periodic (chaotic) window. Obviously, control and anti-control can be realized with mixed switched values for stable and chaotic dynamics.

As can be easily seen, compared to some other chaos control (or anti-control) approaches, the PS algorithm acts without changing the original properties of the synthesized attractor A^o, and this nondestructive technique could be of great value and importance in practical chaos control (anti-control) problems.

4. Stable Behaviors of the Financial System

Now, the above-described PS algorithm is applied to the IVP (2) so as to obtain stable cycles of the system (1) for both integer-order and fractional-order cases. The parameters b and c in system (1) are set to their nominal values, i.e., $b = 0.1$ and $c = 1$.

For this purpose, the underlying IVP is integrated via the standard Runge-Kutta method and the modified GL method (11) for the integer-order and fractional-order cases, respectively. The integration time interval was $I = [0, 400]$ for the integer-order case, and is larger for the fractional-order case due to the more complex dynamics of the latter. In both cases, with few exceptions for the fractional cases, the integration step size was $h = 0.005$.

In order to verify the effectiveness of the PS algorithm, and to compare the attractors A^o and A^*, phase portraits are plotted (red line for A^o and blue line for A^*) and the Hausdorff distance described in [34, Ch 9, p.114] is calculated using the Matlab code [35]. Except a few special cases, the Hausdorff distance is found to be in the order 10^{-4}.

Integer-order financial system

By plotting out the local maxima, the bifurcation diagram of the integer-order financial system (system (1) with $q = 1$) is obtained, as shown in Figure 2. Here, the focus is on the largest stable window centered at $p = 6.5$ (see the enlarged view in Figure 2) and some switched values belonging to the neighboring chaotic windows.

Suppose that one wants to obtain the stable periodic motion (cycle) corresponding to $p^* = 6.5$ based on the switching scheme $[m_1p_1, m_2p_2]$ with $p_1 = 6$ and $p_2 = 7$, both of which exhibit chaotic dynamics (Figure 3 (a) and (b)). In this case, the weights are set to be $m_1 = m_2 = 1$. As shown in Figure 3 (c), one can obtain the synthesized attractor A^o by the PS algorithm, showing a stable cycle which well matches with the averaged attractor A^* that corresponds to $p^* = 6.5$ as given by (10).

It is also possible to change the obtained attractors while the PS algorithm is running on some time interval I. This is demonstrated by applying more than one scheme (9), as shown by the following example. Assuming that, one first applies the same scheme as before, i.e. $[1p_1, 1p_2]$, with $p_1 = 6$ and $p_2 = 7$, it leads to $p_1^* = 6.5$ and A_1^o. Then, if the scheme is changed to $[1p_1, 1p_2]$ with $p_1 = p_1^*$ and $p_2 = 7.38$, another stable cycle A_2^o can be observed, which corresponds to $p_2^* = 6.94$ (see Figure 2). The phase plots of $A_{1,2}^o$ and $A_{1,2}^*$ and the corresponding times series with transients are shown in Figure 4 (a) and (b), respectively. It well demonstrates that the PS algorithm can convert a stable cycle into another one.

Same attractor can be obtained by using a different set of $\mathcal{P}_N$. For example, the PS algorithm with the switching scheme $[1p_1, 1p_2]$, where $p_1 = 5$ (it presents a chaotic attractor as shown in Figure 5 (a)) and $p_2 = 8$ (it presents a stable periodic cycle as shown in Figure 5 (b)), gives the same attractor A^o (Figure 5 (c)) as in the first example above which corresponds to $p^* = 6.5$. Even when p_1 and p_2 are not both from chaotic windows, the PS algorithm still works like a chaos control scheme, yielding the stable attractor A^o.

Another example can be shown by synthesizing the same attractor with $N > 2$ switched values. Specifically, consider the scheme $[m_1p_1, m_2p_2, ..., m_9p_9]$ with $P_9 = \{1, 2, 3, 5, 7, 8, 9, 10, 11\}$ and $m_1 = m_2 = m_3 = 1, m_4 = m_5 = 3, m_6 = m_7 = m_8 = 1, m_9 = 2$. One again obtains $p^* = 6.5$. However, as shown in Figure 6, A^o and A^* are not perfectly matched if $h = 0.005$. This result can be improved by reducing the integration step size, say to $h = 0.001$, and the new result is depicted in Figure 6 (b) for comparison.

As mentioned in Remark 5 ii), the switching scheme in the PS algorithm can also be implemented in a random way. For instance, consider $P_2 = \{6.3, 7\}$ and choose the switching order of p_i, $i \in \{1, 2\}$, randomly in every integration step. Figure 7 depicts the simulation result for $I = [0, 400]$ and $h = 0,002$, which gives $p^* = 6.574...$ as computed by the relation (12). It should be noted that a smaller value of h is used here, due to the random nature of such a switching scheme. As observed from Figure 7, the attractors A^o and A^* are similar, though relative large differences are noted at some points of the attractors. A better matched solution may be found if the time interval I is increased, such that the right-hand side of (12) can better approach the value of p^*.

Remark 6. *As stated in the previous section, the PS algorithm can also* *be utilized as an chaos anti-control scheme. For example, considering the switching scheme* $[1p_1, 1p_2]$ *with* $p_1 = 6.5$ *and* $p_2 = 7.2$*, representing two stable orbits (see Figure 2, 8 (a) and (b)), respectively, the obtained chaotic attractor* A^o *fits the averaged chaotic attractor* A^* *with* $p^* = 6.85$ *(Figure 8 c). It is again noted that the attractors are not matched as good as those in chaos control cases, since chaotic attractors can only be approximated theoretically after a sufficiently long period of time.*

Fractional-order financial system

For the fractional-order setting (system (1) with $q \in (0, 1)$), the modified GL method (11) is adopted.

Compared with the integer case where chaos exists for $q_1 + q_2 + q_3 = 3$, for the fractional variant, as can be seen next, the chaos persists for order less than 3, and the lowest order is $q_1 + q_2 + q_3 = 2.35$ [18]. However, the phenomena observed in integer case, such as period doubling route to chaos, can also be found for the fractional case.

First, consider the commensurate case with $q_1 = q_2 = q_3 = 0.9$. The PS algorithm is applied with $P_2 = \{4.45, 5.45\}$ and the weights $m_1 = m_2 = 1$, thus $p^* = 4.95$. Figure 9 depicts the obtained attractors A^o and A^*, showing a very good matching of these two attractors.

For the incommensurate case, let $q_1 = q_2 = 1$ and $q_3 = 0.9$, while using $P_2 = \{0.5, 0.9\}$ with weights $m_1 = m_2 = 1$ in the PS algorithm. One can verify that $p^* = 0.7$. The stable cycles for both switching and averaged cases are plotted in Figure 10, wherefrom one can easily see that both attractors match very well.

Similarly, the same cycle can be obtained with other switching schemes, such as $[1p_1, 1p_2, 1p_3, 1p_4]$ with $p_1 = 0.5$, $p_2 = 0.6$, $p_3 = 0.8$ and $p_4 = 0.9$.

The simulation result is plotted in Figure 11 for reference.

5. Conclusions

It has been shown, albeit numerically, that a stable cycle of the financial system (1), both integer-order and fractional-order, can be obtained by the parameter switching (PS) algorithm, starting from any set of switching parameter values. The algorithm is also good for obtaining chaotic attractor when a suitable set of switching values is chosen.

The main advantage of the PS algorithm is that it does not affect the underlying stable or chaotic motions of the considered system. In contrast, many other chaos control and anticontrol techniques perform perturbations to the state variables or the parameters of the underlying chaotic system, thereby the obtained stable or chaotic trajectories are no longer the same genuine of the original dynamical system. In this sense, the PS algorithm acts more like as a nondestructive algorithm.

Another advantage of the PS algorithm is that it can provide good explanations to some stable or chaotic motions in nature, which are caused by some apparently very complicate switching dynamics.

Although without further elaboration, the modified GL scheme (11) utilized in this paper appears to be suitable for the problem in interest. Due to a suitable formulation of the solutions obtained by the GL method, sequential computation with reasonable effort becomes realistic. However, the properties of this method deserves further investigation, which is beyond the scope of the present paper but will be carried out in the near future.

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References

References

[1] Smale, S. [1980] The Mathematics of Time: Essays on Dynamical Systems, Economic Processes, and Related Topics, Springer-Verlag, Berlin

[2] Chian, A.C. [2000] Nonlinear dynamics and chaos in macroeconomics, International Journal of Theoretical and Applied Finance, 3(3): 601–601

[3] Feichtinger, G. [1992] Economic Evolution and Demographic Change, Springer, Berlin

[4] Puu, T. [1991] Chaos in duopoly pricing, Chaos Solitons & Fractals, 1(6): 573–581

[5] Fanti L. & Manfredi, P. [2007] Chaotic business cycles and fiscal policy: An IS-LM model with distributed tax collection lags, Chaos Solitons & Fractals, 32(2):736–744, 2007.

[6] Serletis, A. [1996] Is there chaos in economic time series? Canadian Journal of Economics/Revue canadienne d'Economique, 29: 210–212

[7] Lipton-Lifschitz, A. [1999] Predictability and unpredictability in financial markets, Physica D: Nonlinear Phenomena, 133(1-4): 321–347

[8] Goodwin, R. M. [1990] Chaotic Economic Dynamics, Oxford University Press, Oxford

[9] LeBaron, B. [1994] Chaos and nonlinear forecastability in economics and finance, Philosophical Transactions of the Royal Society of London. Series A: Physical and Engineering Sciences, 348(1688): 397–404

[10] Kopel, M. [1997] Improving the performance of an economic system: controlling chaos, Journal of Evolutionary Economics, 7(3): 269–289

[11] Hołlyst, J. A., Hagel, T., Haag, G. & Weidlich, W. [1996] How to control a chaotic economy? Journal of Evolutionary Economics, 6(1): 31–42

[12] Hołyst, J. A. & Urbanowicz, K. [2000] Chaos control in economical model by time-delayed feedback method, Physica A: Statistical Mechanics and its Applications, 287(3): 587–598

[13] Chen, L. & Chen, G. [2007] Controlling chaos in an economic model, Physica A: Statistical Mechanics and its Applications, 374(1): 349–358

[14] Chen, L., Chai, Y. & Wu, R. [2011] Control and synchronization of fractional-order financial system based on linear control, Discrete Dynamics in nature and Society, 2011: 958393

[15] Ma, J. & Chen, Y. [2001a] Study for the bifurcation topological structure and the global complicated character of a kind of nonlinear finance system (I), Applied Mathematics and Mechanics, 22(11): 1240–1251

[16] Ma, J. and Chen, Y. [2001b] Study for the bifurcation topological structure and the global complicated character of a kind of nonlinear finance system (II), Applied Mathematics and Mechanics, 22(12): 1375–1382

[17] Wang, Z., Huang, X. & Shi, G. [2011] Analysis of nonlinear dynamics and chaos in a fractional order financial system with time delay, Computers & Mathematics with Applications, 62(3): 1531–1539

[18] Chen, W. C. [2008] Nonlinear dynamics and chaos in a fractional-order financial system, Chaos, Solitons & Fractals, 36(5): 1305–1314

[19] Abd-Elouahab, M. S., Hamri, N.-E. & Wang, J. [2010] Chaos control of a fractional-order financial system, Mathematical Problems in Engineering, 2010: 270646

[20] Diethelm, K. [2004] The Analysis of Fractional Differential Equations, Springer-Verlag, Berlin

[21] Podlubny, I. [1999] Fractional Differential Equations, Academic Press, San Diego, CA

[22] Kilbas, A. A., Srivastava, H. M. & Trujillo, J. J. [2006] Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, Amsterdam

[23] Baleanu, D., Diethelm, K., Scalas, E. & Trujillo J.J. [2012] Fractional Calculus Models and Numerical Methods (Series on Complexity, Nonlinearity and Chaos. World Scientific)

[24] Bhalekar, S., Daftardar-Gejji, V., Baleanu, D. & Magin, R. [2012] Transient chaos in fractional Bloch equations, Computers and Mathematics with Applications 64(10): 3367–3376

[25] Momani, S., Rqayiq, A.A., & Baleanu, D. [2012] A nonstandard finite difference scheme for two-sided space-fractional partial differential equations, International Journal of Bifurcation and Chaos 22(4): 1–5

[26] Scherer, R., Kalla, S. L., Tang, Y. & Huang, J. [2011] The Grünwald–Letnikov method for fractional differential equations, Computers & Mathematics with Applications, 62(3): 902–917

[27] Li, C. & Zeng, F. [2012] Finite difference methods for fractional differential equations, International Journal of Bifurcation and Chaos, 22(4): 1230014, 28

[28] Mao, Y., Tang, W. K. S. & Danca, M. F. [2010] An averaging model for chaotic system with periodic time-varying parameter, Applied Mathematics and Computation, 217(1): 355–362

[29] Danca, M. F., Romera, M., Pastor, G. & Montoya, F. [2012] Finding attractors of continuous–time systems by parameter switching, Nonlinear Dynamics, 67(4): 2317–2342

[30] Sanders, J. A. & Verhulst, F. [1985] Averaging Methods in Nonlinear Dynamical Systems, Springer, Berlin

[31] Petráš, I. [2011] Fractional-Order Nonlinear Systems, Springer, Berlin

[32] Garrappa, R. [2011] Predictor-corrector PECE method for fractional differential equations, MATLAB Central File Exchange, file ID: 32918

[33] Garrappa, R. [2010] On linear stability of predictor–corrector algorithms for fractional differential equations, International Journal of Computer Mathematics, 87(10): 2281–2290

[34] Falconer, K. J. [2003] Fractal Geometry: Mathematical Foundations and Applications, Wiley, New York

[35] Danziger, Z. [2009] Hausdorff Distance, MATLAB Central File Exchange, File ID: 26738, 2009–2012

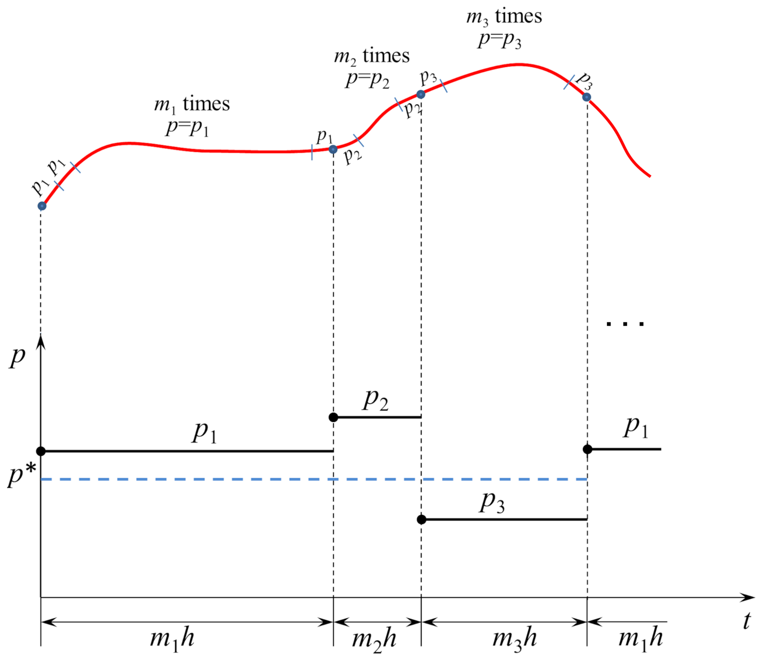


Figure 1: PS algorithm (sketch).

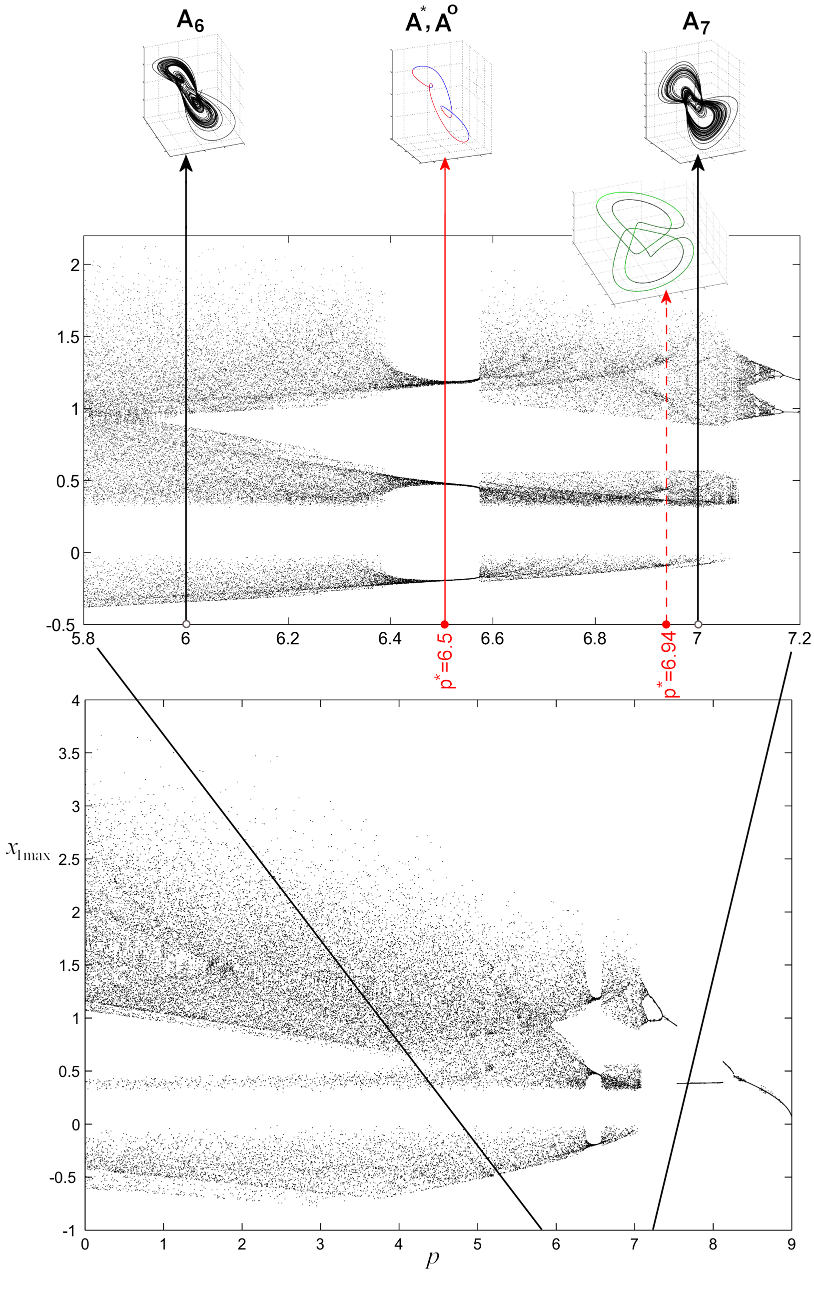


Figure 2: Bifurcation diagram of system (1) for integer case. Local maxima of x_1 are plotted.

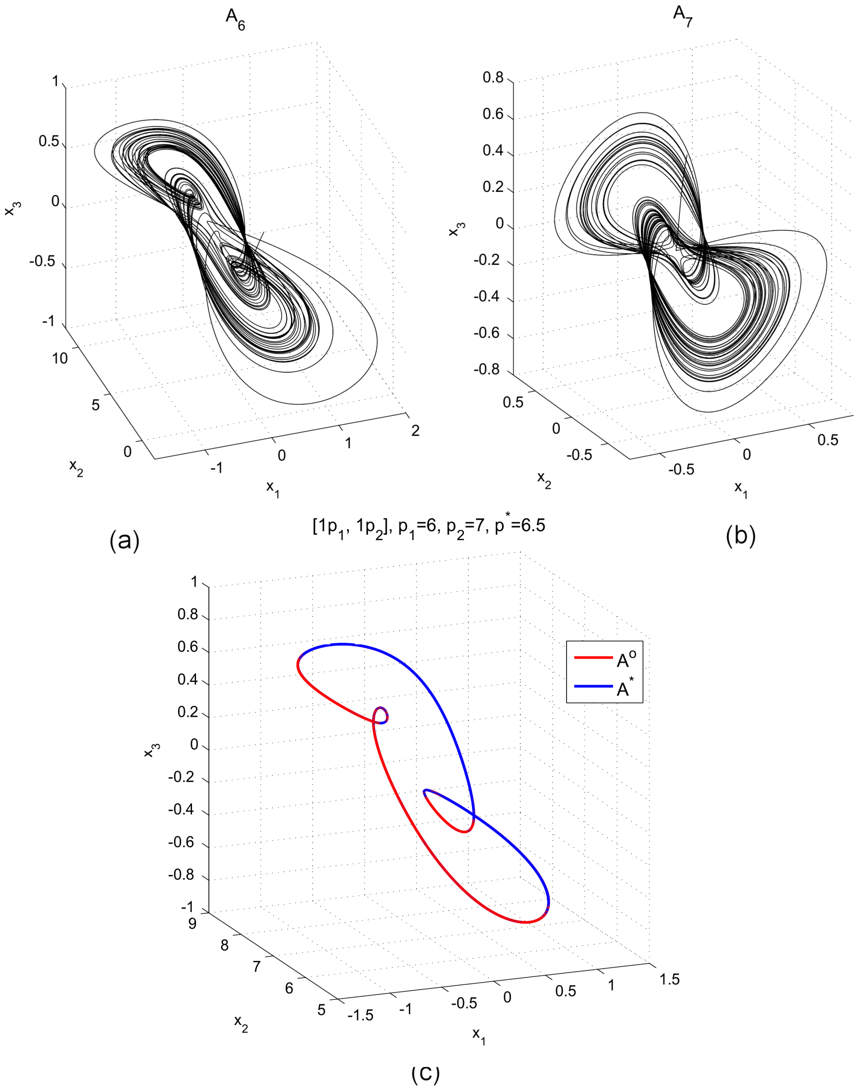
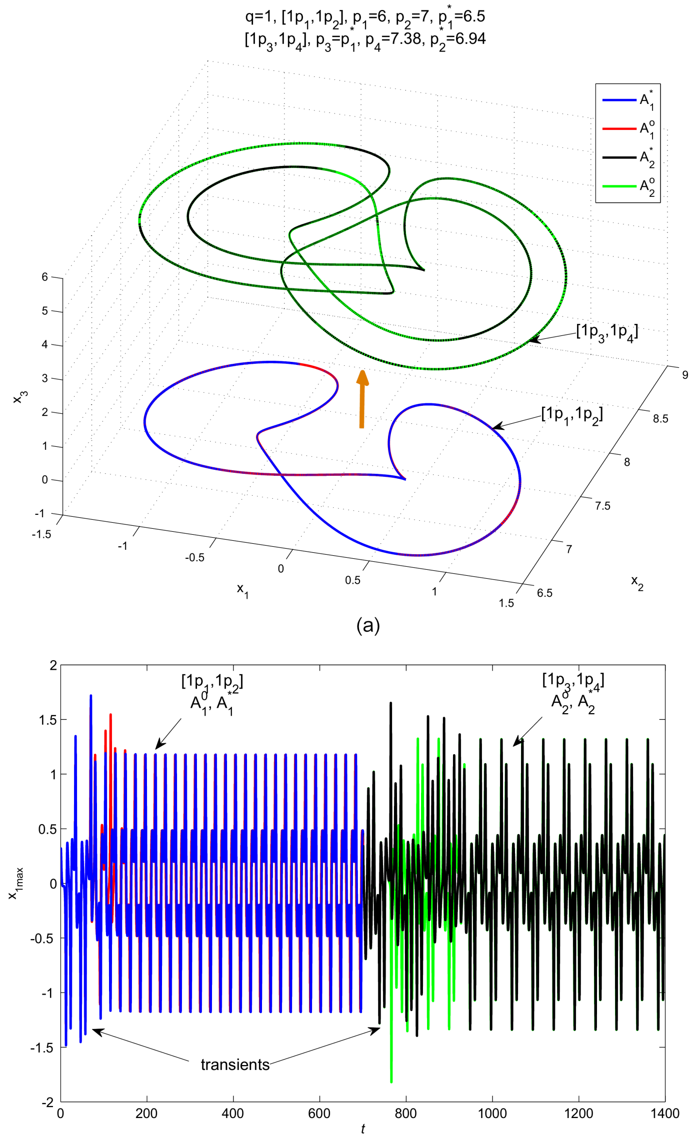


Figure 3: Stable cycle for the integer-order case, corresponding to $p^* = 6.5$ and obtained with the scheme $[1p_1, 1p_2]$ with $p_1 = 6$ and $p_2 = 7$. a), b) The chaotic attractors corresponding to $p_{1,2}$; c) The attractors A^o and A^* over-plotted.



(a)

(b)

Figure 4: PS algorithm applied consecutively with schemes $[1p_1, 1p_2]$ with $p_1 = 6$, $p_2 = 7$ for $t \in [0, 700]$ which gives $p_1^* = 6.5$ and then with $p_1 = p_1^*$ and $p_2 = 7.38$ which gives $p_2^* = 6.94$ for $t \in [700, 1400]$. The attractors A_2^o and A_2^* have been translated for a better view. a) Phase plots of A_1^o, A_1^* and A_2^o, A_2^* respectively; b) Time series.

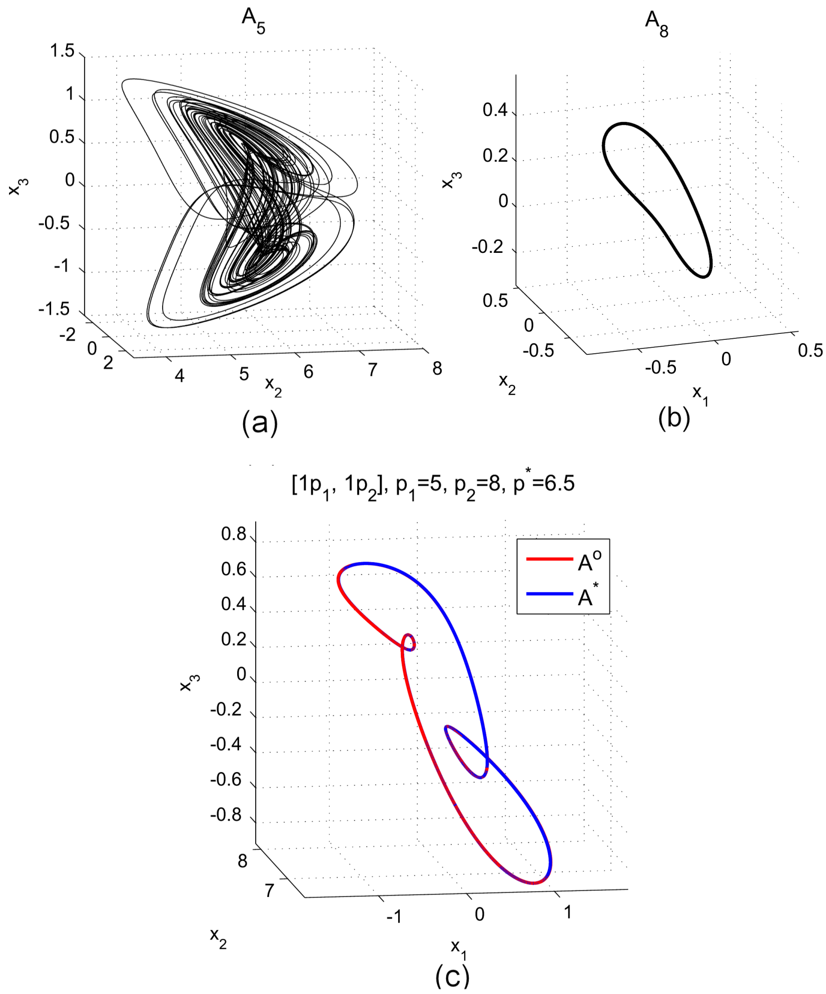


Figure 5: Stable cycle for the integer-order case, corresponding to $p^* = 6.5$ and obtained with the scheme $[1p_1, 1p_2]$ with $p_1 = 5$ and $p_2 = 8$. a), b) The chaotic attractors corresponding to $p_{1,2}$; c) The attractors A^o and A^* over-plotted.

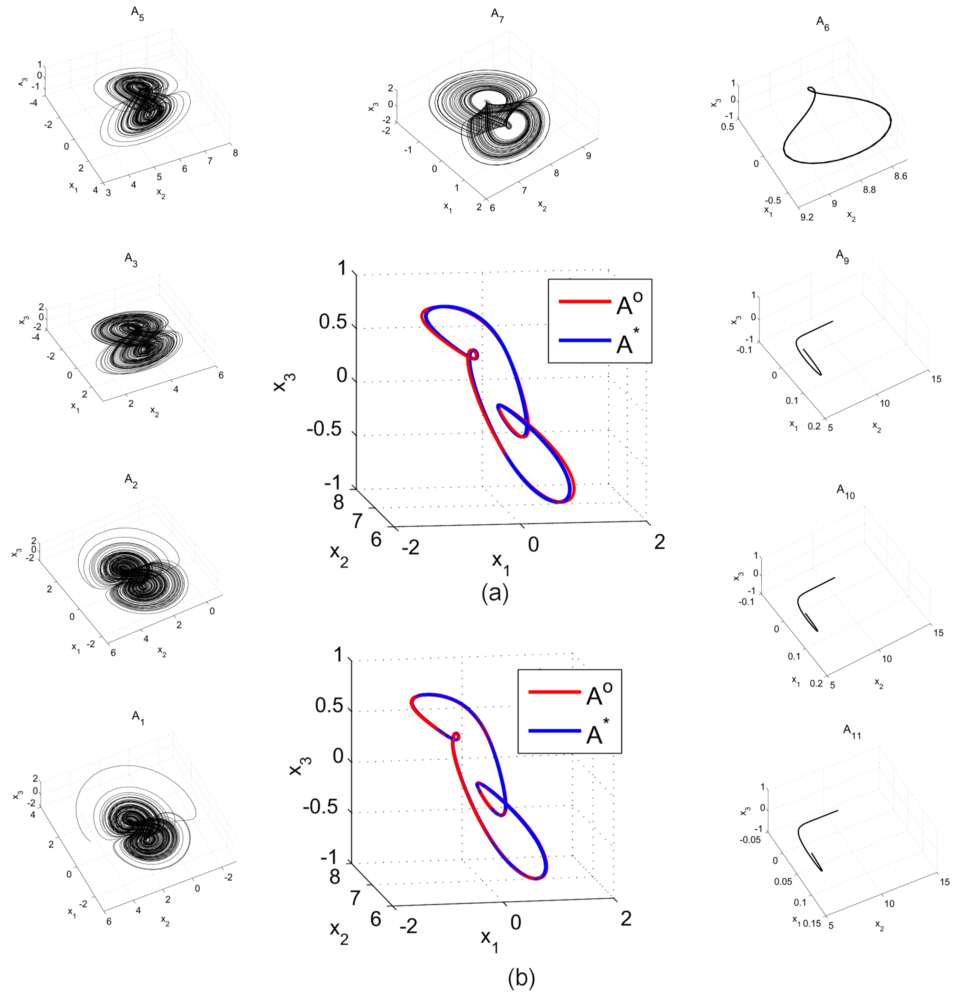


Figure 6: Stable cycle for the integer-order case, corresponding to $p^* = 6.5$ and obtained with the scheme $[1p_1, 1p_2, 1p_3, 3p_4, 3p_5, 1p_6, 1p_7, 1p_8, 2p_9]$ with $P_9 = \{1, 2, 3, 5, 7, 8, 9, 10, 11\}$ (surrounding attractors). a) A^o and A^* obtained with $h = 0.005$; b) A^o and A^* obtained with $h = 0.001$.

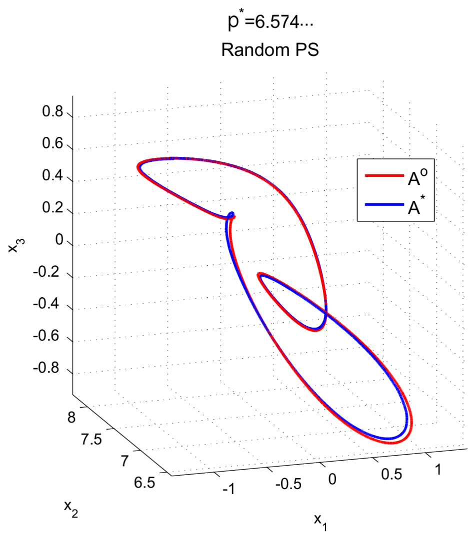


Figure 7: Random PS applied to the values $P_2 = \{6.3, 7\}$ for the integer case, with switching order chosen randomly. The obtained p^* calculated with (12), Remark 5 ii), is $p^* = 6.574...$.

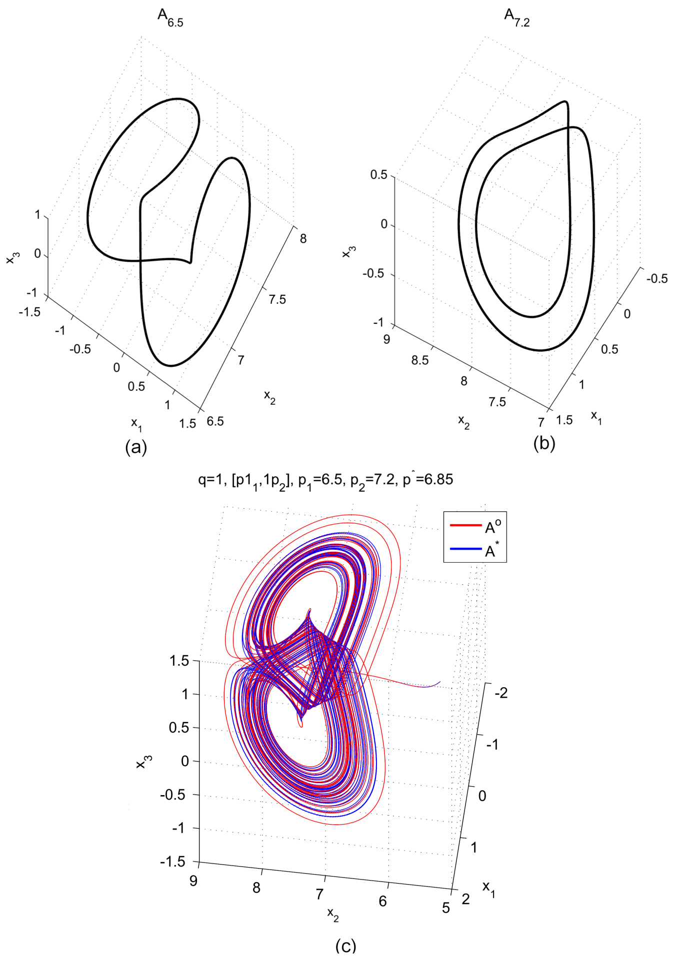


Figure 8: Anti-control of the integer-order financial system obtained with the scheme $[1p_1, 1p_2]$ with $p_1 = 6.5$ and $p_2 = 7.2$. a), b) The cycles corresponding to p_1 and p_2 respectively. c) The synthesized attractor A^o and the average attractor A^*.

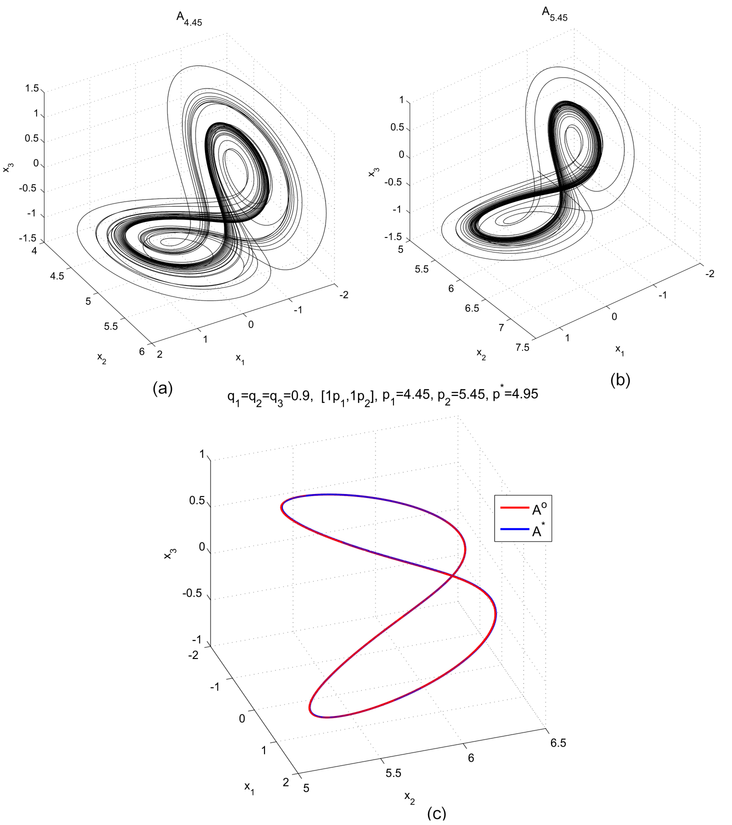


Figure 9: Stable cycle for the fractional-order commensurate case $q_1 = q_2 = q_3 = 0.9$, corresponding to $p^* = 4.95$ and obtained with the scheme $[1p_1, 1p_2]$ with $p_1 = 4.45$ and $p_2 = 5.45$. a), b) The chaotic attractors corresponding to $p_{1,2}$; c) The attractors A^o and A^* over-plotted.

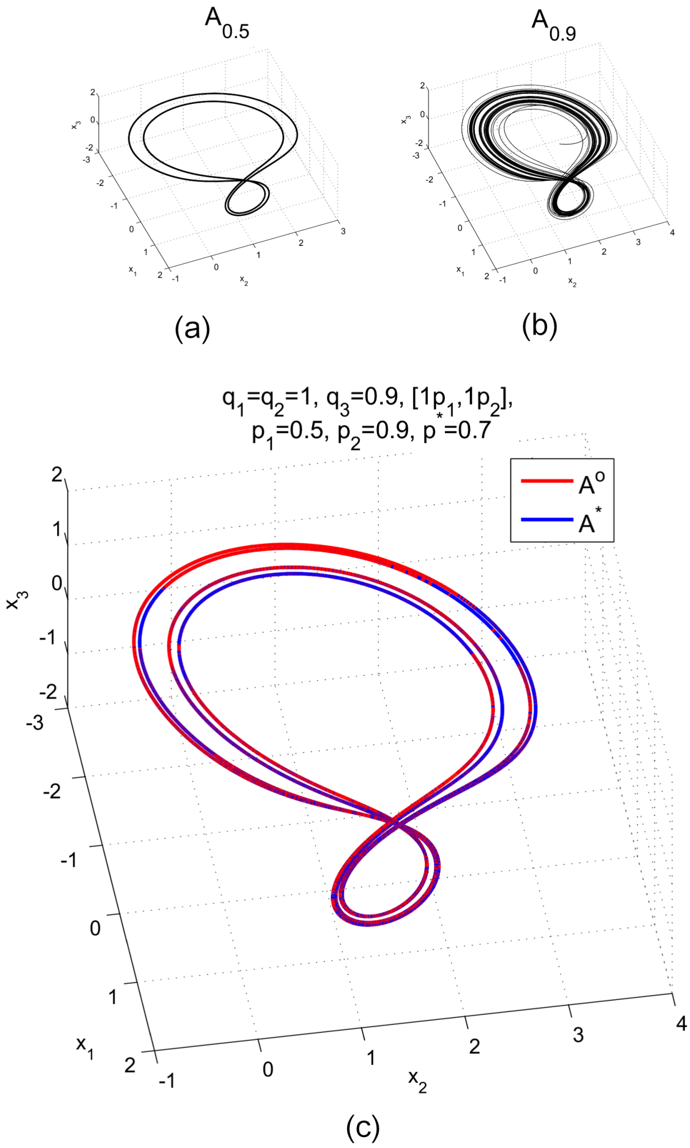


Figure 10: Stable cycle for the non-commensurate case $q_1 = q_2 = 1$ and $q_3 = 0.9$ and corresponding to $p^* = 0.7$ obtained with the scheme $[1p_1, 1p_2]$ with $p_1 = 0.5$ and $p_2 = 0.9$; a) Top: Attractors corresponding to the switched values $p_1 = 0.5$ and $p_2 = 0.9$; b) Bottom: The attractors A^o and A^*.

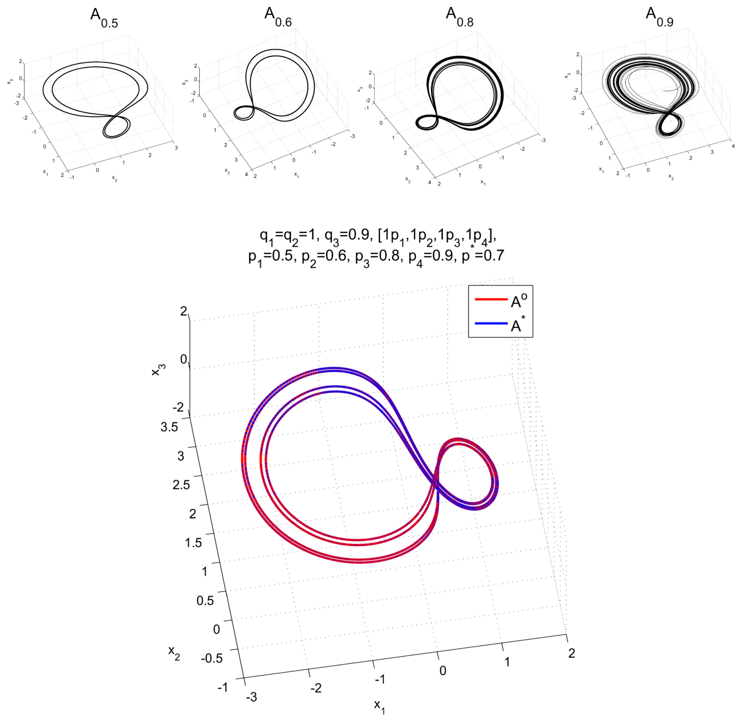


Figure 11: Stable cycle for the non-commensurate case $q_1 = q_2 = 1$ and $q_3 = 0.9$ and corresponding to $p^* = 0.7$ obtained with the scheme $[1p_1, 1p_2, 1p_3, 1p_4]$ with $p_1 = 0.5, p_2 = 0.6, p_3 = 0.8$ and $p_4 = 0.9$; a) Top: Attractors corresponding to the switched values $p_1 = 0.5$, $p_2 = 0.6$, $p_3 = 0.8$ and $p_3 = 0.9$; b) Bottom: The attractors A^o and A^*.