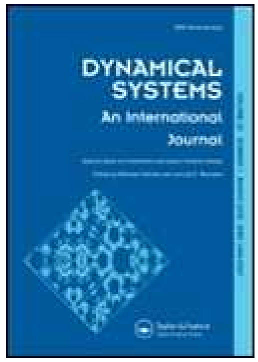

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Stabilizing a class of dissipative dynamical systems by parameter switching

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Abstract

In this paper we prove numerically, via computer graphic simulations, that switching the control parameter of a dynamical system belonging to a class of dissipative continuous dynamical systems, one can obtain a stable attractor. In this purpose, while a fixed step-size numerical method approximates the solution to the mathematical model, the parameter control is switched every few integration steps, the switching scheme being time periodic. The switch is made inside of a considered set of admissible parameter values. Moreover we show that the obtained *synthesized* attractor belongs to the class of all admissible attractors for the considered system and matches to the *averaged* attractor obtained for the control parameter replaced with the averaged switched parameter values. The synthesis algorithm may force the system to evolve along on a stable attractor whatever the parameter values are switched and introduces a convex structure inside of the attractor set via a bijection between the set of parameter control values and the attractors set. The synthesis algorithm beside his utility in systems stabilization, when some desired parameter control cannot be directly accessed, may serve as model for the dynamics encountered in reality or experiments e.g. three-species food chain models, electronic circuits etc. This method compared to the OGY algorithm, where only small perturbations of parameter control can be issued, allows relatively large parameter perturbations.

The present work extends the results obtained by us previously and is applied to Lorenz, Rőssler and Chen systems.

keywords: stable attractors, chaotic attractors, dissipative dynamical systems

Mathematics Subject Classification: 37C50, 37D45

1 Introduction

Until recently, the preferred approach to chaotic dynamical systems has been to avoid them. In recent years one of the main goals of nonlinear science is to deal with chaos and to prevent it. A number of chaos control algorithms have been proposed and the aspects of controlling chaos have been investigated

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the most known method, OGY, being developed by Ott, Grebogi, and Yorke [1] which takes advantage of the sensitivity to initial conditions, and other characteristics of chaotic systems, to produce periodic behavior from chaotic behavior. As it is known, unstable periodic orbits of nonlinear dynamical system that possesses a chaotic attractor, are typically dense. Thus, there exists a large number of periods and the system can be stabilized in many different hyperbolic periodic orbits. Ott, Grebogi and Yorke have shown that chaotic systems may he controlled by making only small changes in an accessible control parameter. Thus, once the control parameters have been calculated it is necessary to wait for the system to enter a region of state space, then to issue small modifications to the control parameter to ensure that the system will come back to the same controllable region.

If the control methods pay careful attention to the (generally small) control parameter changes, nontrivial questions arise: what happens if, during the system evolution, the control parameter is simply switched between few values in a time deterministic or even random manner? Does his qualitative chaotic or stable structure change? If for the considered control parameter values the system evolves chaotic, can these switches stabilize the system, or reversely, if the system is stable for the considered values, the parameter switches could enhance chaos?

The first answers to these questions was given in [2] the present paper having the goal to extend the study of this algorithm (the *synthesis algorithm*). Thus, while in [2] we presented a general parameter switching scheme to synthesize attractors (stable or chaotic), in this paper we prove, via numerical experiments, that switching in a time deterministic manner the control parameter, a system belonging to a class of dissipative systems may be forced to evolve along any desired stable periodic attractor whatever the considered number of parameters chosen to be switched is considered and whatever the original behavior was: stable or chaotic.

The counter arguments to the argument that this "control" technique would be useless since we can directly set the system to be stable by choosing an appropriate parameter value, are the following: from a practical design point of view, it is sometimes difficult or even impossible to generate a specific attractor by a particular parametric value on a physical device, also especially in natural systems, deterministic (or even random) time switches of the control parameter may natural happen and have a realistic meaning in e.g. ecological systems and circuitry.

Compared to the existing family of OGY methods which require detailed calculations, while the attractor is generated via discrete perturbations applied to the control parameter, the synthesis algorithm implies only a priori parameter switching scheme.

Remark 1 *While all the known control methods work on unstable attractors, the synthesis algorithm cannot be considered as a typical* control *or* stabilization algorithm *-which stabilize a periodic unstable orbit- since if before to start the algorithm the system can evolve stable and then the algorithm just change the behavior from a stable attractor to another one. Therefore this algorithm could be viewed as a kind of "general stabilization" because either the system evolves initially stable or chaotic, the algorithm may "switch" this behavior to another one. Also the synthesis algorithm can be use to chaotify [2], [3] and may explain why in real systems chaos appear.*

The class of chaotic systems considered here can be represented by a continuous-time autonomous dissipative model depending linearly on a single real bifurcation parameter, expressed in the general form of the following Initial Value Problem:

$$S: \ \dot{x} = f_p(x), \quad x(0) = x_0, \tag{1}$$

where f_p is an $\mathbb{R}^n$-valued function with a bifurcation parameter $p \in \mathbb{R}$ and $n \geq 3$, and has the expression

$$f_p(x) = g(x) + pAx, \tag{2}$$

in which $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous-time nonlinear function, A is a real constant $n \times n$ matrix, $x_0 \in \mathbb{R}^n$, and $t \in I = [0, \infty)$.

The aims of this paper is to prove numerically, via computer graphic simulations, that the synthesis algorithm can determine the system S to evolve along a stable (limit-cycle) attractor, whatever the initial behavior was and, to verify that the obtained attractor belongs to the set of all admissible attractors of S.

This class of dynamical systems is enough large containing known systems such as Lorenz, Rőssler, Chen.

Throughout, the existence and uniqueness of solutions are assumed, and it is supposed that there exist only hyperbolic equilibria.

Despite the fact that there are differences between computation and theory, nonetheless, the numerical integration of (1)-(2) via computer graphic simulations can give excellent approximations to the orbits within the invariant sets [4]. Also, the orbits that start near a hyperbolic attractor will stay near and they will be shadowed by orbits within the attractor because attractors arise as the limiting behavior of orbits. Therefore, the shadowing property of hyperbolic sets [5] enables us to recover long time approximation properties of numerical orbits.

The algorithm presented in this paper consists in use a time varying, or more preciously, periodically switching parameter, according to some design rule. It will be demonstrated, empirically by various experiments, that a desired attractor can be duly obtained by the proposed switching scheme.

The organization of the paper is as follows. Section 2 presents the synthesis algorithm, while in Section 3 the algorithm is applied to obtain a desirable stable orbit. Three examples are considered.

2 Switching scheme

Notation 2 *Let $\mathcal{A}$ be the set of all global attractors [6] depending on parameter p, including attractive stable fixed points, limit cycles and chaotic (possibly strange) attractors; $\mathcal{P} \subset \mathbb{R}$ be the set of the corresponding admissible values of p and $\mathcal{P}_N = \{p_1, p_2, \ldots, p_N\} \subset \mathcal{P}$ a finite ordered subset of $\mathcal{P}$ containing N different values of p, which determines the set of attractors $\mathcal{A}_N = \{A_{p_1}, A_{p_2}, \ldots, A_{p_N}\} \subset \mathcal{A}$.*

The assumption that we can access all the values of $\mathcal{P}_N = \{p_1, p_2, \ldots, p_N\}$ for which the system is stable and/or chaotic is assumed to held.

Due to the assumed dissipativity, $\mathcal{A}$ is non-empty and it follows naturally that for the considered class of systems, a bijection may be defined between the sets $\mathcal{P}$ and $\mathcal{A}$. Thus, giving any $p \in \mathcal{P}$, a unique global attractor is specified, and vice versa.

Remark 3 *Different orbits may have different attractors. Also it is a particularly, not uncommon, situation when the attractor is the same for all orbits. Because in this paper, computer simulations are used as the major analytical tool, the ω-*limit set *(actually, its approximation [7]) is considered after neglecting a sufficiently long period of transients. In other words, by* attractors *(background on attractor notion can be found in [8]) it is suitable to understand here the ω-*limit set *obtained by a numerical method for ODEs with fixed step size h after the transients were neglected.*

Choosing a finite subset $\mathcal{P}_N = \{p_1, p_2, \ldots, p_N\}$ the deterministic synthesis algorithm relies on the following deterministic time switching rule which is applied indefinitely

$$[(m_1 h)p_{\varphi(1)},\ (m_2 h)p_{\varphi(2)}, \ldots, (m_N h)\ p_{\varphi(N)}], \tag{3}$$

where the weights m_i are some positive integers and φ permutes the subset $\{1, 2, \ldots, N\}$. The algorithm acts as follow: for the first time subinterval of length $m_1 h$, p will have the value $p_{\varphi(1)}$, for the next m_2 integration steps, $p = p_{\varphi(2)}$ (Figure 1) and so on until the N-th time subinterval of length $m_N h$ where $p = p_{\varphi(N)}$ then the algorithm repeats. Thus, the relation (3) is $(m_1 + m_2 + ... + m_N)h$ periodic because it repeats (3) in each successive time intervals $\left[\left(\sum_{k=1}^{N} m_k h\right) i, \left(\sum_{k=1}^{N} m_k h\right)(i+1)\right]_{i=0,1,2,\ldots}$. In order to simplify the notation, for a fixed step size h, the scheme (3) will be denoted

$$[m_1 p_{\varphi(1)},\ m_2 p_{\varphi(2)}, \ldots, m_N\ p_{\varphi(N)}]. \tag{4}$$

For example the scheme $[2p_3, 3p_1, 1p_2]$ represents the infinite sequence of $p : 2p_3, 3p_1, 1p_2, 2p_3, 3p_1, 1p_2, \ldots$ which means that while the considered numerical method integrates (1)-(2), p switches in each $m_i h$ time subinterval.

The *synthesized* attractor will be denoted hereafter A^*. To prove that $A^* \in \mathcal{A}$ we prove that A^* is identical to the *averaged* attractor denoted A_{p^*} with

$$p^* = \frac{\sum_{k=1}^{N} p_{\varphi(k)} m_k}{\sum_{k=1}^{N} m_k}. \tag{5}$$

If we denote $\alpha_k = m_k / \sum_{k=1}^{N} m_k$ it is easy to see that p^* is a convex combination $p^* = \sum_{k=1}^{N} \alpha_k p_{\varphi(k)}$, since $\sum_{k=1}^{N} \alpha_k = 1$. Therefore $p^* \in (p_1, \ldots, p_N)$, whatever the values p_i are chosen.

Also, taking into account the bijection between $\mathcal{P}$ and $\mathcal{A}$, we are entitled to consider that the same convex structure is preserved into $\mathcal{A}$. Therefore for whatever switched values of p in some subset $\mathcal{P}_N$, the synthesized attractor A^* will belong inside the ordered set $\mathcal{A}_N$, with the order induced by the mentioned bijection.

Remark 4 *i) The initial conditions play an important role since for a specific value $p \in \mathcal{P}$ there is a single global attractor but which could be composed by several local attractors. For example for the Lorenz system for $p = 2.5$ there are three local attractors: the origin (saddle) and two symmetrical fixed points (sinks) $X_1, 2 \pm 2, \mp 2, 1.5)$. Also In some cases, the unique local attractor may also be the global one (e.g. for $p = 28$, there exists only a single local attractor, which is a global attractor too, the Lorenz strange attractor). Therefore, the computer simulations for A^* and A_{p^*} are considered from the same initial conditions. This choice is obvious because of the possible coexistence of several attraction basins when the global attractor contains several local attractors.*
ii) The size of the integration step h is an important parameter which may influence the results due the convergence properties of the considered method for ODEs. Obviously, both attractors A^* and A_{p^*} are simulated with the same fixed step size.

Let us consider the case of Lorenz system

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1), \\ \dot{x}_2 &= x_1(p - x_3) - x_2, \; , \\ \dot{x}_3 &= x_1 x_2 - c x_3, \end{aligned} \tag{6}$$

with the known parameters setting $a = 10$ and $c = 8/3$, p being the control parameter.
Using the scheme $[2p_1, 3p_2, 2p_3, 4p_4, 3p_5]$ with $p_1 = 125$, $p_2 = 130$, $p_3 = 140$, $p_4 = 144$, $p_5 = 220$, $\varphi(i) = i, \; i \in \{1, \ldots 5\}$, the synthesis attractor A^* is depicted in Figure 2(a). In Figure 2(b) A^* and the averaged attractor A_{p^*} with p^* given by (5) $p^* = (2p_1 + 3p_2 + 2p_3 + 4p_4 + 3p_5) / (2 + 3 + 2 + 4 + 3) = 154$ are overplotted. The histogram (Figure 2(c)) underlines this identity[1]. To better understand the way in which the algorithm behaves, the bifurcation diagram of the maximum state variable x_1 is referred (Figure 3) wherefrom it may be noticed that despite the fact that only one attractor, A_{p_5}, is stable and the other four are chaotic, the obtained attractor A^* is stable. Actually, this could happen even if all considered attractors are chaotic or, reversely, choosing all attractors stable, function of the weights m, the synthesized attractor A^* may be chaotic. This property, resulted from convexity property, may be noted in above example if instead $p_5 = 220$, we chose $p_5 = 166$ (Figure 4) when the synthesized attractor A^* is chaotic and is identical to the averaged attractor A_{p^*} with $p^* = 142.428$, In Figure 5, A^*, A_{p^*} are plotted beside histogram and Poincaré section (with the plane $x_3 = 135$) which underline the matching between the two attractors. Thus, for a dynamical system modeled by (1)-(2) using the switching scheme (4) whatever is $\mathcal{P}_N$, for proper choice of the weights m the synthesized attractor A^* always belongs to $\mathcal{A}$. To relative large values for m or large N correspond larger differences between the two attractors A^* and A_{p^*} (see the detail in Figure 12(c)). However A^* still remains inside of an acceptable small neighborhood of A_{p^*}.

The synthesis algorithm with the scheme (4) is depicted in Figure 6

$$\begin{array}{l}
Input: \; N, \; T_{\max}, \; h, \; m_1, \ldots, m_N, \; p_{\varphi(1)}, \ldots p_{\varphi(N)} \\
repeat \\
\qquad p = p_{\varphi(1)} \\
\qquad for \; i = 1 \; to \; m_1 \; do \\
\qquad\qquad integrate \; (1) - (2) \\
\qquad\qquad t = t + h \\
\qquad end \\
\qquad \vdots \\
\qquad p = p_{\varphi(N)} \\
\qquad for \; i = 1 \; to \; m_N \; do \\
\qquad\qquad integrate \; (1) - (2) \\
\qquad\qquad t = t + h \\
\qquad end \\
until \; t \geq T_{\max}
\end{array}$$

Figure 6

[1] This identity for the chaotic attractors case can be considered only "asymptotic", since they are fully depicted only after an infinity time. This relative difference between the two attractors, A^* and A_{p^*}, can be remarked in Figure 3 (more details on identitity notion may be found in [2])

To support the identity between A^* and A_{p^*} which shows the fact that A^* is an attractor belonging to $\mathcal{A}$, histograms and Poincaré sections besides the phase plots were drawn after the transients were neglected. Also the Haussdorf distance between the two trajectories was calculated (see Appendix)

3 Synthesized stable attractors

Let us consider the system modeled by (1)-(2) and the sets $\mathcal{P}_N$ and $\mathcal{A}_N$. Suppose that for some objective reasons certain targeted value(s) of $\widehat{p} \notin \mathcal{P}_N$ -for which the system behaves stable- cannot be accessible. Then, using the bifurcation diagram it is possible to synthesize a stable orbit corresponding to $\widehat{p}$ using the synthesis algorithm with the values of $\mathcal{P}_N$. The only sufficient (and necessary too [2]) condition on $\widehat{p}$ is that it belongs to the inside of the real closed interval $[p_1, \ldots, p_N]$ (because of convexity property, $\widehat{p}$ cannot be chosen outside of $[p_1, \ldots, p_N]$).To synthesize the attractor $A_{\widehat{p}}$, we must choose m. Having $p_1, \ldots, p_N$ fixed, the equation (5) for $p^* = \widehat{p}$ with the unknown m has to be solved. Then, with the obtained values for m, the scheme (4) is applied. The synthesized attractor A^* is identical, as shown above, to $A_{\widehat{p}}$. Thus using the synthesizes algorithm one can force the system to evolve on the desired stable orbit corresponding to $\widehat{p}$.

Also the following situation is possible: $\mathcal{P}_N$ is not set a priori. Then, both unknowns, m and the set $\mathcal{P}_N$, have to be determined such (5) to be verified with the only known $\widehat{p}$. Obvious, the solutions are not unique in both situations because the elements of $\mathcal{P}_N$ belong in a compulsory way to one of the infinite number of p-intervals which compose $\mathcal{P}$.

For example, suppose that we want to synthesize, with the scheme $[m_1 p_2, m_2 p_1]$, a stable orbit for the Lorenz system corresponding to $\widehat{p} = 160$ (see the bifurcation diagram in Figure 3) starting from $\mathcal{P}_N = \{100, 190\}$ Then a solution to (5) is $m_1 = 2$ and $m_2 = 1$.

Remark 5 *i) The switching scheme may be applied even in a random way to stabilize or chaoticize the behavior of a system modeled by (1)-(2). Thus, starting from $\mathcal{P}_N$, whatever values for m are taken and for a random path in (3), because of the convexity, p^* will belongs to the open interval $(p_1, \ldots, p_N)$ [3].*
ii) This algorithm may serve in some cases as explanation of stabilization/chaoticisation of a real system, where accidentally the parameter are switched in a deterministic or random way.
iii) The strategy to choose the best scheme (4) depends on the given system and especially on the accessible values for p.

Examples

The algorithm is applied next to generate stable periodic attractors of three known systems: Lorenz system, Chen system

$$\begin{aligned} \dot{x}_1 = &\quad a(x_2 - x_1), \\ \dot{x}_2 = &\quad (p - a)x_1 - x_1 x_3 + p x_2, \\ \dot{x}_3 = &\quad x_1 x_2 - b x_3, \end{aligned}$$

with $a = 35$ and $b = 3$, and Rössler system

$$\begin{aligned} \dot{x}_1 = &\quad -x_2 - x_3, \\ \dot{x}_2 = &\quad x_1 + a x_2, \\ \dot{x}_3 = &\quad b + x_3(x_1 - p), \end{aligned}$$

with $a = b = 0.1$. Everywhere p is considered as control parameter.

Computational graphics tools such as three dimensional phase portraits and histograms for x_1 were plotted. In order to check the difference between the orbits of A^* and A_{p_N} Hausdorff distance d_H was calculated (see Appendix) the highest value being obtained for Rössler system due to the sensitivity of the computed results to the integration time steps as pointed out in [9]. For all considered cases $d_H(A^*, A_{p_N})$ is of order of $10^{-3} \div 10^{-2}$ which represents a good identity between the orbits. The numerical method to integrate (1)-(2) was the standard Runge-Kutta with the step size $h = 10^{-4} \div 10^{-2}$ function of the integrated system.

For the sake of simplicity only the cases $N = 2, 3$ are presented. The results are depicted in Table 1.

System	*Scheme*	p_1	p_2	p_3	p^*	*Graphical results*
Lorenz	$[1p_1, 2p_2]$	81.5	98	–	92.5	*Figure* 7
	$[2p_1, 3p_2, 3p_3]$	125	140	175	149.375	*Figure* 8
Chen	$[1p_1, 1p_2]$	25.916	26.25	–	26.083	*Figure* 9
	$[7p_1, 3p_2, 4p_3]$	23	24	32	25.7857	*Figure* 10
Rössler	$[2p_1, 1p_2]$	6	12.3	–	8.1	*Figure* 11
	$[2p_1, 5p_2, 3p_3]$	18	25	31	25.4	*Figure* 12

Table 1

Better synthesis results are observed in Chen and Lorenz systems, while some small derivation is noticed in the case of the Rössler system because of the above mentioned sensitivity.

4 Conclusions and further directions

In this paper we proved numerically that while a numerical method solves the system modeled by (1)-(2) we switch periodically the control parameter following a determinist rule, the orbit can be forced to reach a desired stable attractor. The control parameter is switched inside a finite subset $\mathcal{P}_N$ of the set of all admissible values $\mathcal{P}$, every finite time subintervals. The synthesized stable attractor are identical to an attractor, belonging to the set of all attractors $\mathcal{A}$, which corresponds to the averaged value of the switched values of $\mathcal{P}_N$.

Further mathematical studies on this algorithm remain a task for future works such as the convergency of the synthesized attractor to the averaged attractor, the study of the step size influence and so on.

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Appendix

The Hausdorff distance in a metric space, or Hausdorff metric, measures how far two compact non-empty subsets of the considered metric space are from each other.

The classical (symmetrical) Hausdorff distance between two (finite) sets of points, A and B, is defined as ([10])

$$d_H(A,B) = \max\left\{\sup_{x\in A}\ \inf_{y\in B}\ d(x,y),\ \sup_{y\in B}\ \inf_{x\in A}\ d(x,y)\right\}.$$

The Hausdorff distance between two curves is defined as the maximum distance to the closest point between two curves. If the curves are defined as the sets of ordered pair of coordinates $A = \{a_1, a_2, \dots, a_n\}$, $B = \{b_1, b_2, \dots, b_m\}$, then the distance to the closest point between a point a_i to the set B is

$$d(a_i, B) = \min_j \|b_j - a_i\|.$$

Thus, the Hausdorff distance is

$$d_H(A,B) = \max\left\{\max_i\{d(a_i,B)\}, \max_j\{d(b_j,A)\}\right\}.$$

Figure Captions

Figure 1: Time parameter partition (sketch)

Figure 2: The synthesize algorithm applied for Lorenz system with the scheme $[2p_1, 3p_2, 2p_3, 4p_4, 3p_5]$ with $p_1 = 125$, $p_2 = 130$, $p_3 = 140$, $p_4 = 144$, $p_5 = 220$; a) The synthesized attractor A^* b) the averaged attractor A_{p^*} with $p^* = 154$;c) Histogram.

Figure 3: The bifurcation diagram for Lorenz system illustrating the scheme $[2p_1, 3p_2, 2p_3, 4p_4, 3p_5]$ with p values as in Figure 2

Figure 4: The bifurcation diagram for Lorenz system illustrating the scheme $[2p_1, 3p_2, 2p_3, 4p_4, 3p_5]$ with p values as in Figure 2 except $p_5 = 166$.

Figure 5: The synthesize algorithm applied for Lorenz system with the scheme $[2p_1, 3p_2, 2p_3, 4p_4, 3p_5]$ with $p_1 = 125$, $p_2 = 130$, $p_3 = 140$, $p_4 = 144$, $p_5 = 220$; a) The synthesized attractor A^* b) the averaged attractor A_{p^*} with $p^* = 154$;c) Histogram; d) Poincaré section with the plane $x_3 = 135$.

Figure 7: The synthesize algorithm applied for Lorenz system with the scheme $[1p_1, 2p_2]$ with $p_1 = 81.5$, $p_2 = 98$; a-b) Attractors A_{p_1} and A_{p_2}; c) A^* and A_{p^*} overplotted with $p^* = 92.5$; c) Histogram;

Figure 8: The synthesize algorithm applied for Lorenz system with the scheme $[2p_1, 3p_2, 3p_3]$ with $p_1 = 125$, $p_2 = 140$ and $p_3 = 175$; a-c) Attractors A_{p_1}, A_{p_2} and A_{p_3}; d) A^* and A_{p^*} overplotted with $p^* = 149.375$; e) Histogram;

Figure 9: The synthesize algorithm applied for Chen system with the scheme $[1p_1, 1p_2]$ with $p_1 = 25.916$, and $p_2 = 26.25$; a-b) Attractors A_{p_1} and A_{p_2}; c) A^* and A_{p^*} overplotted with $p^* = 26.083$; d) Histogram;

Figure 10: The synthesize algorithm applied for Chen system with the scheme $[7p_1, 3p_2, 4p_3]$ with $p_1 = 6$, $p_2 = 24$ and $p_3 = 32$; a-c) Attractors $A_{p_1}$and A_{p_2} ; d) A^* and A_{p^*} overplotted with $p^* = 25.7857$; e) Histogram;

Figure 11: The synthesize algorithm applied for Rössler system with the scheme $[2p_1, 1p_2]$ with $p_1 = 125$, and $p_2 = 12.3$; a-b) Attractors A_{p_1} and A_{p_3}; c) A^* and A_{p^*} overplotted with $p^* = 8.1$; d) Histogram;

Figure 12: The synthesize algorithm applied for Rössler system with the scheme $[2p_1, 5p_2, 3p_3]$ with $p_1 = 18$, $p_2 = 25$ and $p_3 = 31$; a-c) Attractors $A_{p_1}$and A_{p_2} ; d) A^* and A_{p^*} overplotted with $p^* = 25.4$; e) Histogram;

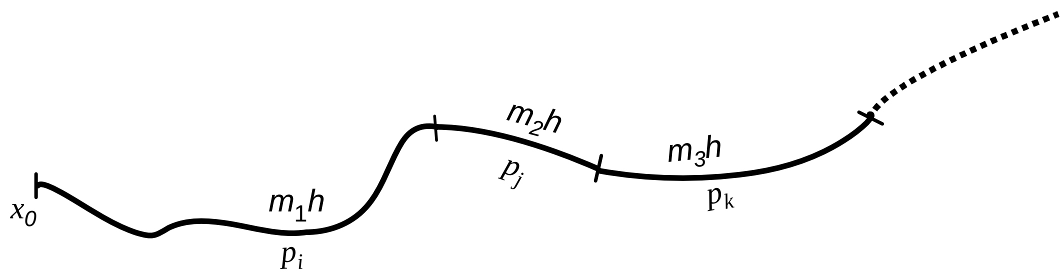


Figure 1

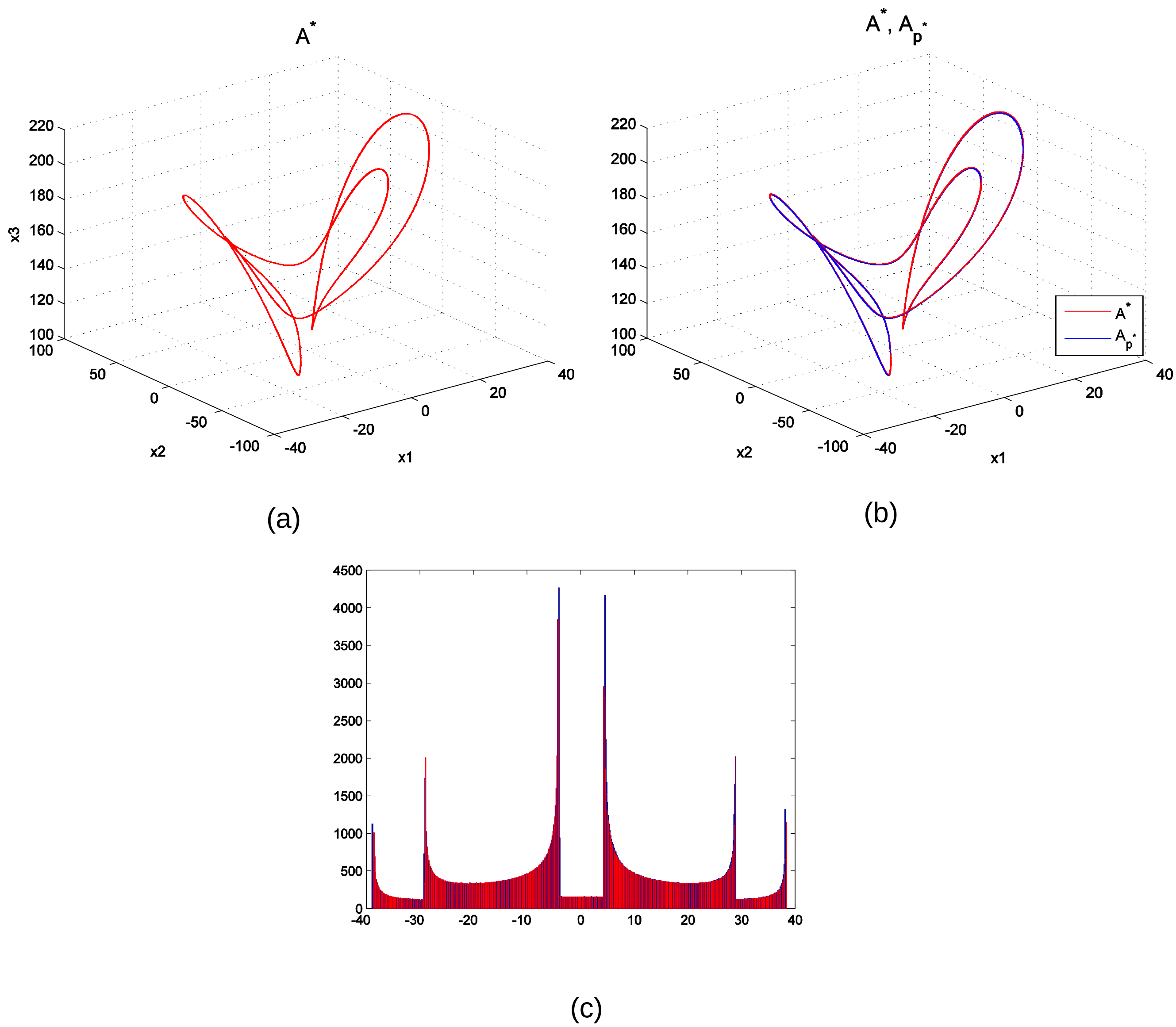
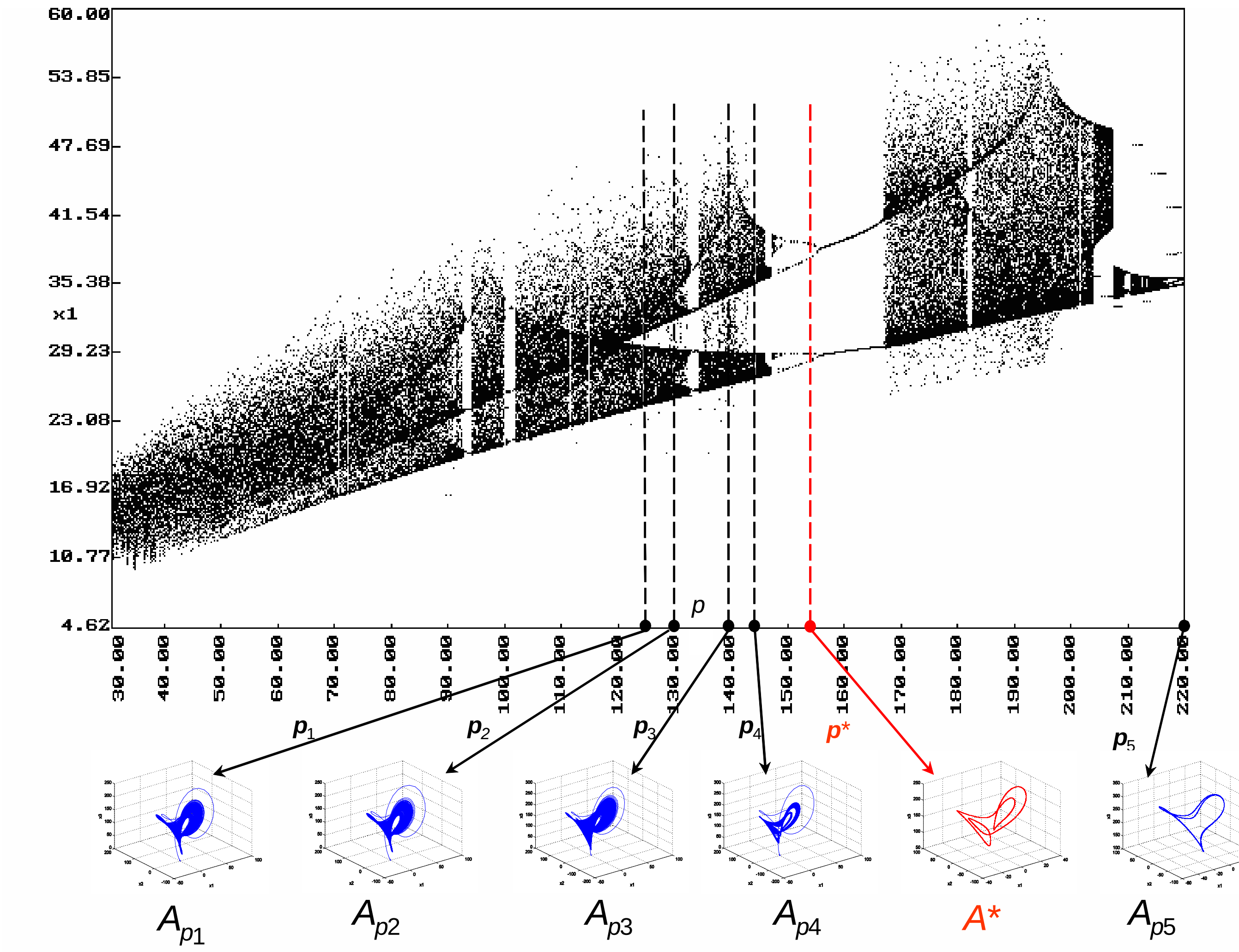


Figure 2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53.85
47.69
41.54
35.38
x1
29.23
23.08
16.92
10.77
4.62
p
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40.00
50.00
60.00
70.00
80.00
90.00
100.00
110.00
120.00
130.00
140.00
150.00
160.00
170.00
180.00
190.00
200.00
210.00
220.00
p_1
p_2
p_3
p_4
p^*
p_5
A_{p1}
A_{p2}
A_{p3}
A_{p4}
A^*
A_{p5}

Figure 3

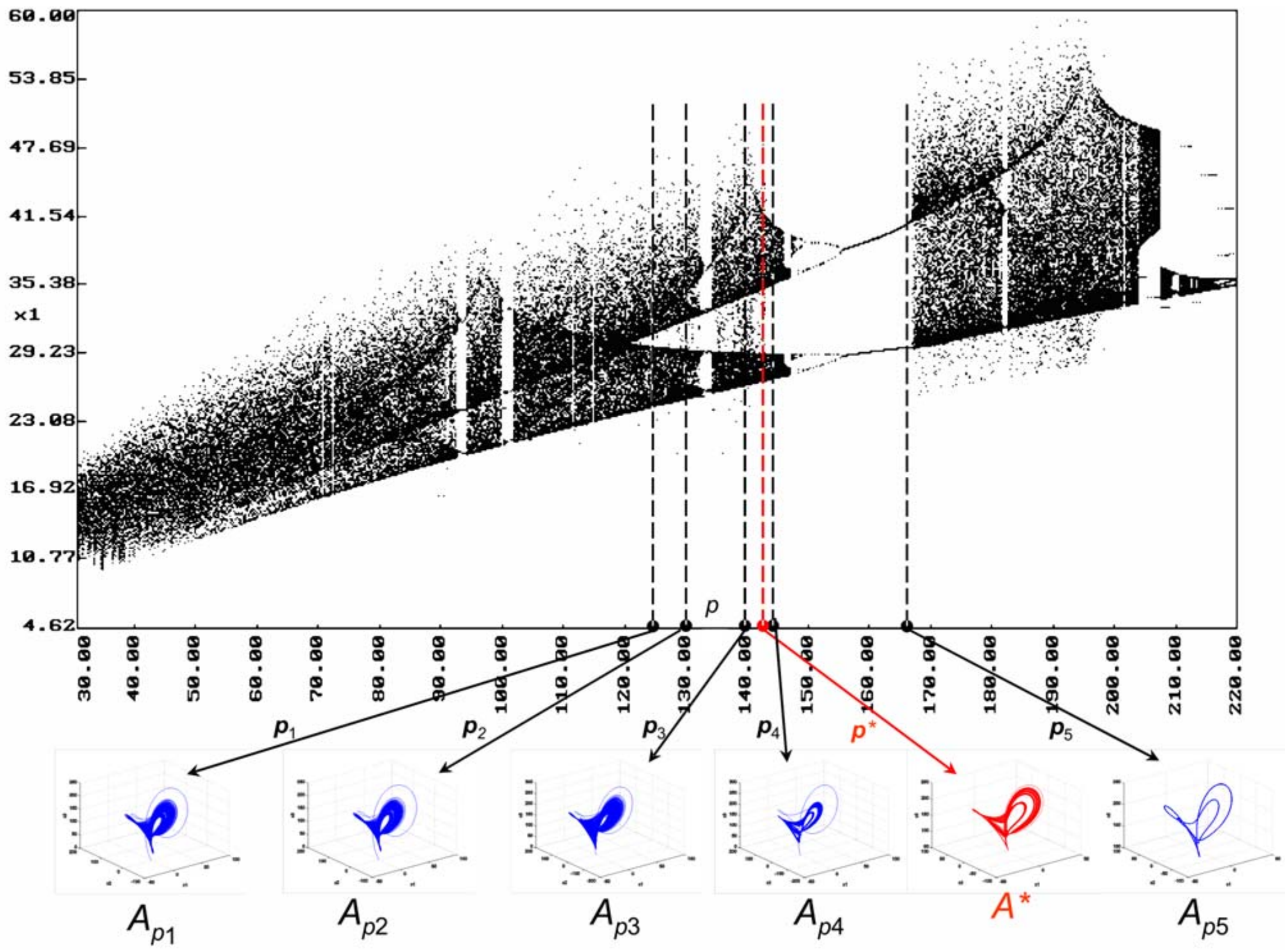
60.00
53.85
47.69
41.54
35.38
x1
29.23
23.08
16.92
10.77
4.62
p
30.00
40.00
50.00
60.00
70.00
80.00
90.00
100.00
110.00
120.00
130.00
140.00
150.00
160.00
170.00
180.00
190.00
200.00
210.00
220.00
p_1
p_2
p_3
p_4
p^*
p_5
A_{p1}
A_{p2}
A_{p3}
A_{p4}
A^*
A_{p5}

Figure 4

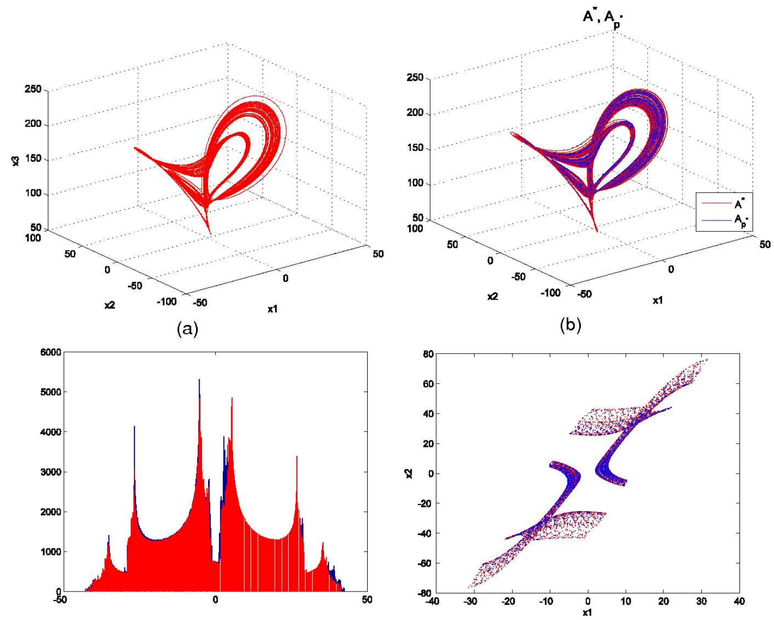

Figure 5

216x279mm (600 x 600 DPI)

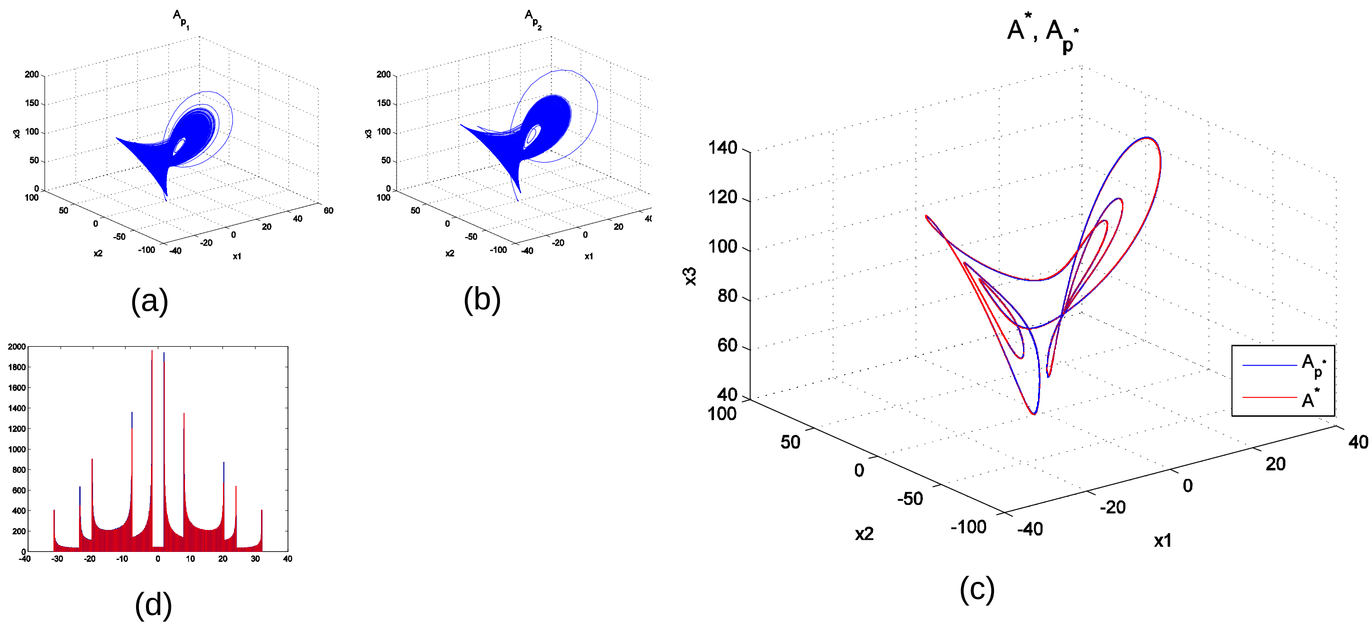

Figure 7

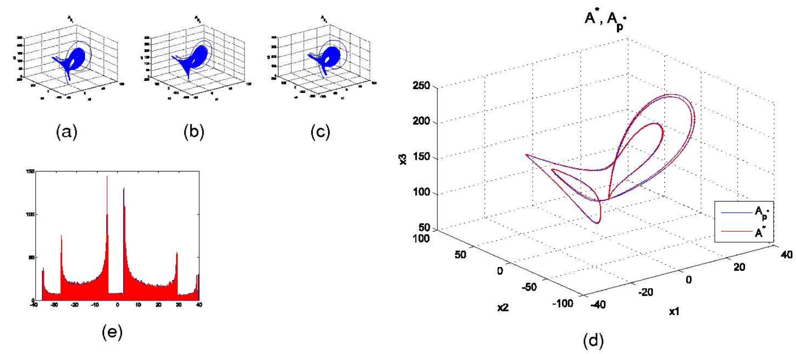

Figure 8

216x279mm (600 x 600 DPI)

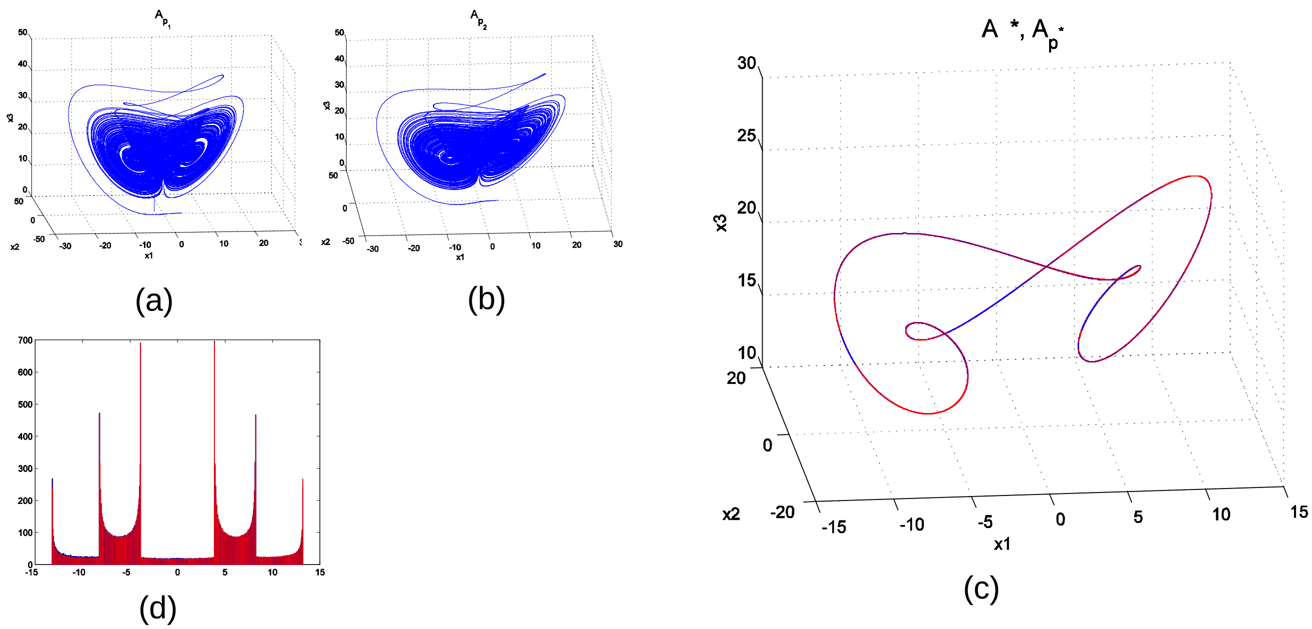

Figure 9

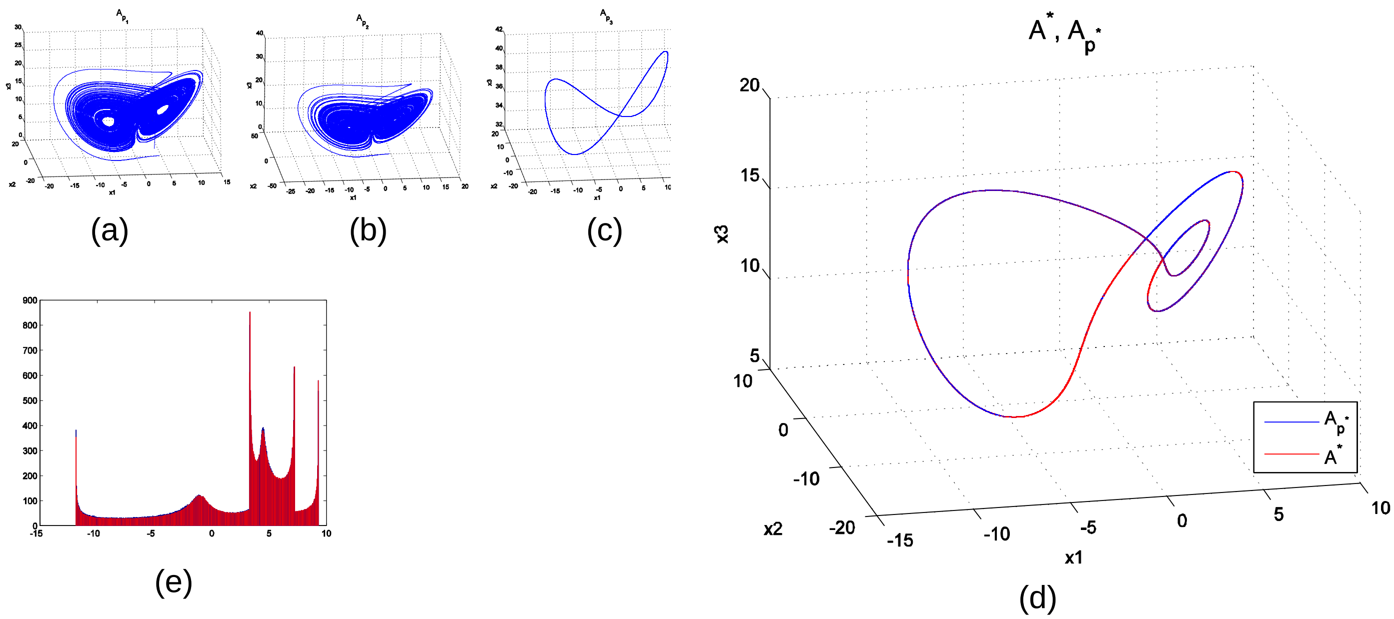

Figure 10

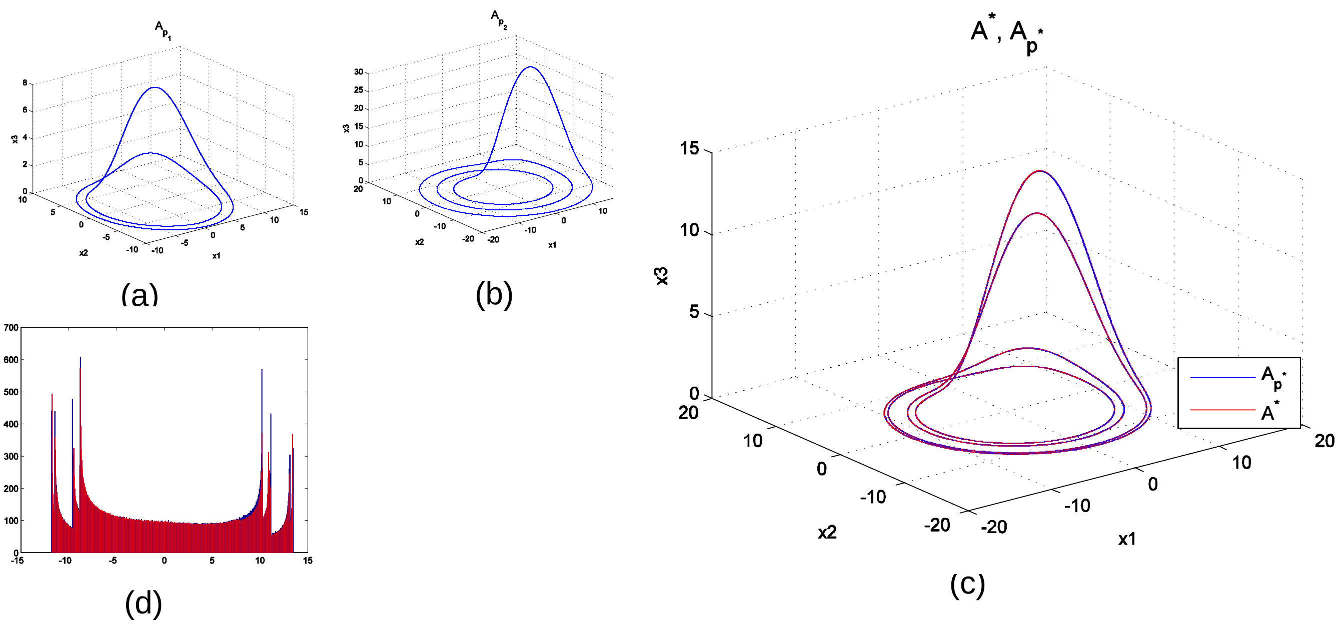

Figure 11

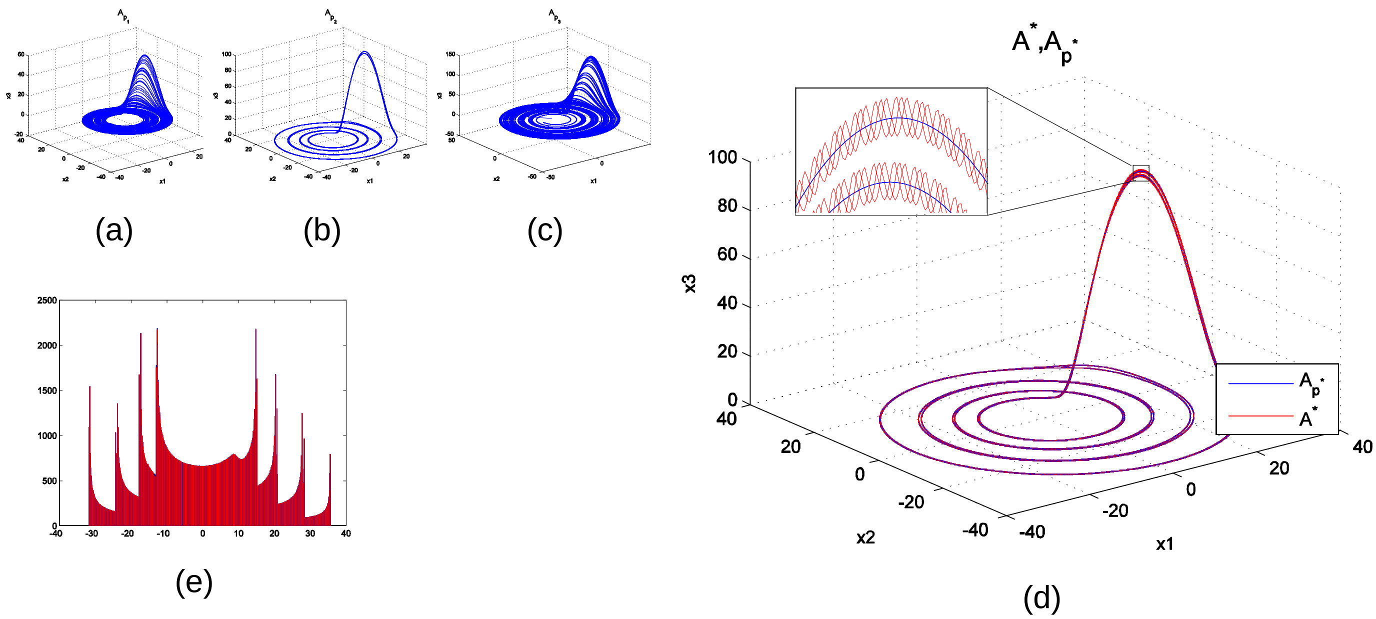

Figure 12