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Detailed Analysis of a Nonlinear Prey-predator Model

MARIUS DANCA[1], STELIANA CODREANU[2] and BOTOND BAKÓ[2]
[1]*Technical University, Cluj-Napoca, Cluj-Napoca, Romania*
[2]*Babeş-Bolyai University, Department of Theoretical Physics, Cluj-Napoca, Romania*

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Abstract. A model of competition between populations of two species, described by a two dimensional map, is analytically and numerically studied. A rich dynamics is observed.

Key words: Deterministic chaos, Two-dimensional mapping, Strange attractor, Hopf bifurcation, Prey-predator model, Nonlinear dynamics.

1. Introduction

The problem of the competition between populations of two species is an old one and different models have been proposed to understand its mechanism. The well known prey-predator model due to the mathematician Volterra [1] has the form of a system of differential equations:

$$\begin{aligned} \dot{x} &= ax - bxy, \\ \dot{y} &= -cy + dxy, \end{aligned} \tag{1}$$

where x and y represent the number of the prey and of the predators, respectively, at time t, and a, b, c, d are positive parameters. This model considers that in the absence of predators the number of prey grows exponentially and also that in the absence of a prey population, the number of predators decreases exponentially. The terms $(-bxy)$ and $(+dxy)$ describe the prey-predators encounters which are favorable to predators and fatal to prey. If one considers some harvesting activity, the model can be changed as following (M. Martelli [2]):

$$\begin{aligned} \dot{x} &= ax - bxy - \gamma x, \\ \dot{y} &= -cy + dxy - \gamma y, \end{aligned} \tag{2}$$

and the result is that a moderate harvesting activity favors the prey population. From the model (1), Basykin [3] has created a novel approach to describe a self-oscillatory system:

$$\dot{x} = ax - b\frac{xy}{1 + \varepsilon x} - \beta x^2,$$

$$\dot{y} = -cy + d\frac{xy}{1+\varepsilon x}, \tag{3}$$

where the parameter ε determines the limitation of the growth velocity of the predator population with the increase of the number of prey. The parameter β takes into account the limitation of the prey population due to mutual competition.

There are also the others prey-predator models which consider a number of interacting species larger than two [4] or which assume a parasitic infection of the populations [5, 6].

Another possible way to understand the complex problem of the competition between two interacting species is by using discrete models. Such a model with considers the prey's growth be governed by a logistic map is the following two dimensional map:

$$x_{n+1} = ax_n(1-x_n) - bx_ny_n,$$

$$y_{n+1} = dx_ny_n, \tag{4}$$

where a, b and d are nonnegative parameters.

The aim of this work is a detailed study of this model. We present, by using both analytic and numerical methods, the domains of the values of the parameters when the system has stable stationary states, the conditions for Hopf bifurcations and the chaotic domain. Some numerical examples which illustrate these behaviours are presented too.

2. The Behavior of the System

The bidimensional map under consideration has two fixed points (the equilibrium points):

$$X_1(0,0) \quad \text{and} \quad X_2\left(\frac{1}{d}, \frac{a}{b}\left(1-\frac{1}{d}\right) - \frac{1}{b}\right). \tag{5}$$

A study of the stability of these fixed points is performed with the usual linearised stability technique. Thus, if we note generally the fixed point with $X(x^*, y^*)$ and set $x_{n+1} = x^* + x'_{n+1}$ and $y_{n+1} = y^* + y'_{n+1}$, we obtain from Equation (4):

$$\begin{pmatrix} x'_{n+1} \\ y'_{n+1} \end{pmatrix} = \begin{pmatrix} a - 2ax^* - by^* & -bx^* \\ dy^* & dx^* \end{pmatrix} \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}. \tag{6}$$

Then the stability of the fixed point is established from the roots of the corresponding eigenvalue equation:

$$\det[J - \lambda I] = 0, \tag{7}$$

where J is the matrix of the transformation (6).

For the fixed point X_1, Equation (7) has the roots $\lambda_1 = 0$ and $\lambda_2 = a$. Thus X_1 is stable for the values $a < 1$ and for any $(b, d) \in \mathbf{R}_+$.

For the second fixed point, X_2, the roots of Equation (7) are:

$$\lambda_{1,2} = \left(1 - \frac{a}{2d}\right) \pm \frac{1}{2}\sqrt{\left(\frac{a}{d} + 2\right)^2 - 4a}. \tag{8}$$

Both eigenvalues are real for (λ_R) and $|\lambda_{1,2}| < 1$ if

$$\left(\frac{a}{d} + 2\right)^2 > 4a \quad \text{and} \quad \frac{a}{d} + 1 < a, \tag{9}$$

which implies:

$$\begin{aligned} & d \in \left(\frac{a}{a-1}\,,\ \frac{a}{2(\sqrt{a}-1)}\right], \\ & a > 1. \end{aligned} \tag{10}$$

The eigenvalues $\lambda_{1,2}$ become complex (λ_C) and are inside the unit circle in the complex λ-plane for

$$\left(\frac{a}{d} + 2\right)^2 < 4a \quad \text{and} \quad a < \frac{2a}{d} + 1, \tag{11}$$

with

$$\begin{aligned} & d \in \left[\frac{a}{2(\sqrt{a}-1)}\,,\ \frac{2a}{a-1}\right), \\ & a > 1. \end{aligned} \tag{12}$$

The conditions (10) and (12) determine the domains of the values of parameters a and d for which the second fixed point, X_2, is stable.

For

$$\left(\frac{a}{d} + 2\right)^2 < 4a \quad \text{and} \quad a = \frac{d}{d-2}, \tag{13}$$

the eigenvalues

$$\lambda_{1,2} = \tfrac{1}{4}[(5 - a) \pm i\sqrt{10a - a^2 - 9}]. \tag{14}$$

Let us consider a complex $\lambda_{1,2}(a)$ in the form:

$$\lambda_{1,2}(a) = A(a)[\cos\,\alpha(a) \pm i\,\sin\,\alpha(a)], \tag{15}$$

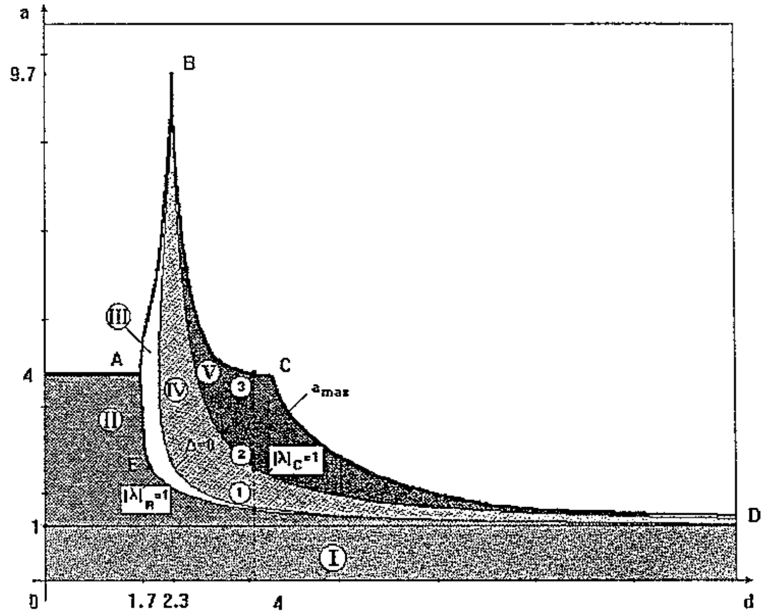

Figure 1. Diagram of domains of the values of the parameters which correspond to different behaviors of the system.

where

$$A(a) = \sqrt{a - 2a/d} \quad \text{and} \quad \alpha(a) \in (0, \pi).$$

For

$$a = \frac{d}{d-2} = a^*, A(a^*) = 1 \quad \text{and} \quad \left.\frac{dA}{da}\right|_{a=a^*} > 0 \quad \text{if } d > 2.$$

From Equation (14) we have:

$$\lambda_{1,2} = \tfrac{1}{4}[(5 - a^*) \pm i\sqrt{10a^* - a^{*2} - 9}],$$

whence

$$\alpha(a^*) = \arctan \frac{\sqrt{4d-9}}{2d-5}. \tag{16}$$

If $\alpha(a^*)/\pi$ is irrational or $\alpha(a^*)/\pi = 2m/n$, with m and n relatively prime $(n \neq 3, 4)$, then for $a = a^*$ there is a Hopf bifurcation. The phase trajectories are attracted by an elliptical invariant curve (when $\alpha(a^*)/\pi$ is irrational) or by a periodic trajectory (when $\alpha(a^*)/\pi = 2m/n$). In the case $n = 3$ or $n = 4$, the bifurcation picture is more complicated. For example, at $a^* = d = 3, \alpha(3) = \arctan\sqrt{3}$ and

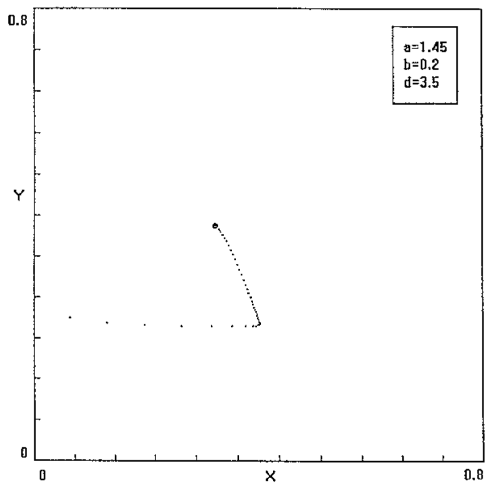

Figure 2. An attractor fixed point.

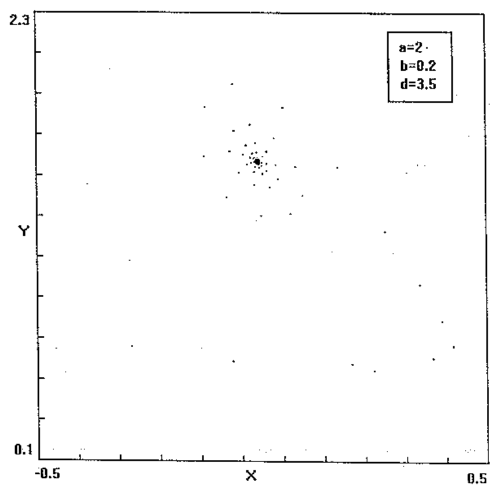

Figure 3. A stable fixed point.

$\alpha(3)/\pi = 3$, whence $2m/n = 3$ with $m = 1$ and $n = 6$. The phase trajectories close to X_2, for values of a and d slightly larger then 3, are attracted by a periodic orbit of period 6.

For the values of parameters a and d larger than those for which the Hopf bifurcation are observed, the system has a complex (chaotic) behavior.

We note that the values of the parameter b determine only the position of the fixed point X_2 in the phase plane, and also that when $y = 0$ the system has logistic behavior.

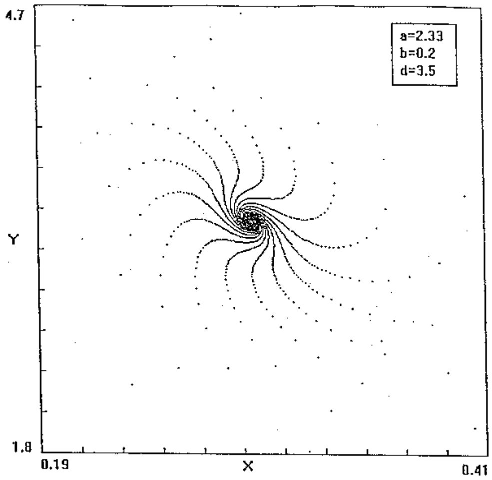

Figure 4. Phase portrait of the map before a Hopf bifurcation.

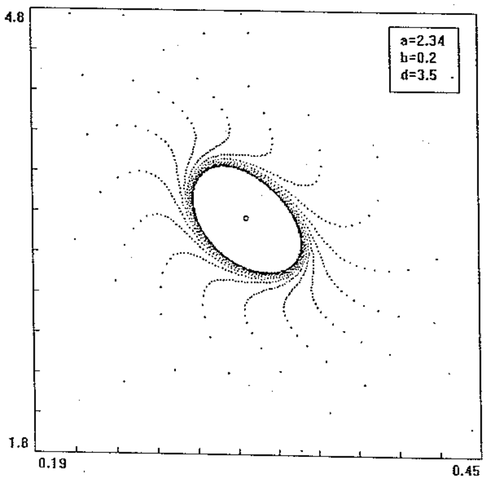

Figure 5. Phase portrait of the map after a Hopf bifurcation.

3. Numerical Investigations

In Figure 1 we present a diagram of the domains of the values of parameters a and d which correspond to the conditions analytically established for the existence and stability of fixed points and for Hopf bifurcations. The domain I corresponds to the values of a and d for which X_1 is the single attractor of the system. This fixed point degenerates in the segment [0,1] of the x-axis for values of parameters in domain II. The curve AED represents the inferior limit of the domain of parameters for which the second fixed point X_2 exists.

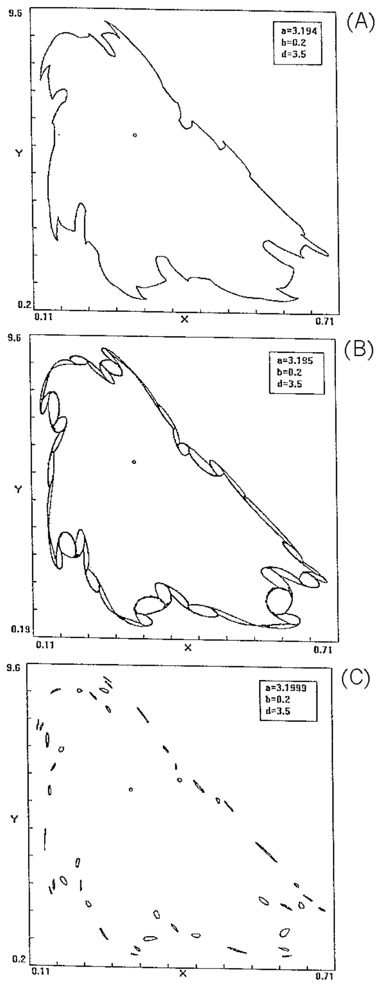

Figure 6. The loss of stability of a limit cycle for three successive values of parameter a.

The arc AE was numerically determined, but ED was obtained from the condition $|\lambda_R| = 1$ where $\lambda_R = \lambda_{1,2} \in \mathbf{R}_+$. The domain III corresponds to the values of parameters for which $|\lambda_R| < 1$. The curve $\Delta = 0$ delimits the domains of parameters where the roots of eigenvalue equation are real $(\lambda_{1,2} = \lambda_R)$ and complex $(\lambda_{1,2} = \lambda_C)$, respectively. In the domain IV, the parameters correspond to λ_C inside the unit circle in the complex λ-plane. The curve $|\lambda_C| = 1$, where the eigenvalues intersect this unit circle, was obtained from the condition $a = d/(d-2)$ when Hopf bifurcations are possible. The domain of existence of X_2 is inside the curve ABCDEA, for which the maximum values of $a(a_{\max})$ were numerically determined. Actually this domain has an asymptotic behavior at $a = 1$ when

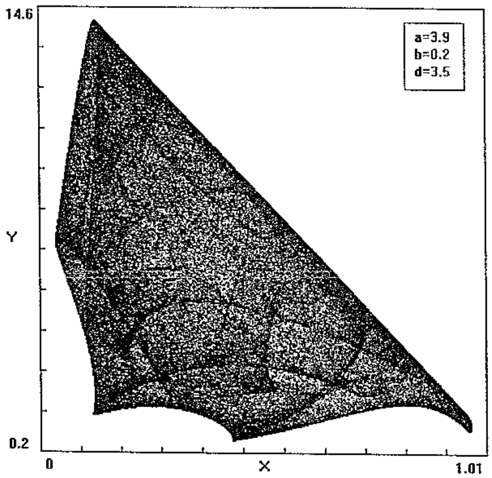

Figure 7. A strange attractor of the map.

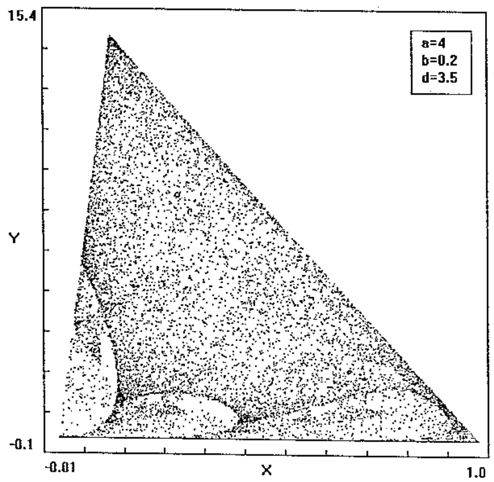

Figure 8. The attractor is a segment of x-axis.

$d \to \infty$. For some values of parameters in the region V, the limit cycles, obtained after Hopf bifurcations, become unstable and the system seems to be chaotic.

To illustrate the behavior of the system for values of parameters of different domains presented above, we chose $d = 3.5, b = 0.2$ and $a \in (0, 4)$ (the dotted vertical line on the Figure 1). For example, for values of a in the proximity of the point labeled 1 in Figure 1, and which belong to the domains III and IV, the fixed point X_2 is a stable attractor (see Figures 2 and 3). The point labeled 2 on Figure 1 is a Hopf bifurcation point. In Figures 4 and 5, the behavior of the system before and after bifurcation is shown.

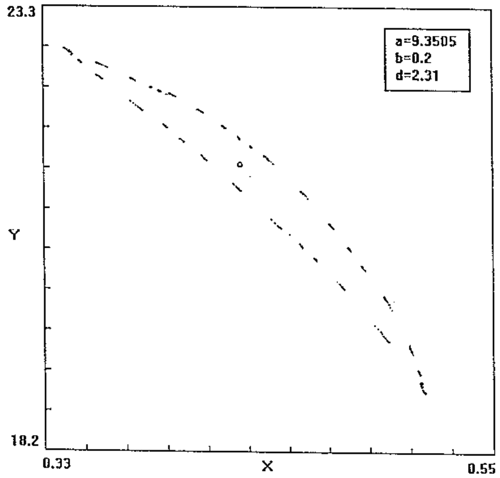

Figure 9. A limit cycle which degenerates into many attractor points.

The loss of stability of a limit cycle for a value of parameter a which belongs to domain V is illustrated in Figure 6.

In Figure 7, we show a strange attractor. The system has a chaotic behavior in the proximity of the point labeled 3 in Figure 1.

For $a = 4$ the strange attractor is destroyed and the system has a logistic-like behavior. This is shown in Figure 8 where actually the attractor is a segment of the x-axis. All the points with $x \neq 0$ and $y \neq 0$ in this figure are transients.

For $a > 4$ and arbitrary d, the phase trajectories of the system become infinite. But for $a > 4$ and $d \approx 2.31$, the numerical simulations have a great sensitivity to the choice of the initial conditions and we noticed that there are limit cycles with very small basins of attraction. For example in Figure 9 is depicted such a limit cycle which degenerates into many attractor points. In this case we chose an initial condition very close to the X_2, which is inside a small limit cycle. This limit cycle is stable inside and unstable outside.

4. Conclusions

In the above analysis of the considered prey-predator model we have shown by using analytic and numerical methods that the system can exhibit a rich behavior. We have determined the domains of the values of parameters (a, d) for which the system has stationary states or chaotic behavior. We have also established the values of parameters when Hopf bifurcations are possible.

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Address for correspondence: Dr Steliana Codreanu, Department of Theoretical Physics, Babeş-Bolyai University, 3400 Cluj-Napoca, Romania

Fax: +40 64 191906; e-mail: scodr@hera.ubbcluj.ro