Controlling chaos in discontinuous dynamical systems

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Abstract

In this paper we consider the possibility to implement the technique of changes in the system variables to control the chaos introduced by Güemez and Matías for continuous dynamical systems to a class of discontinuous dynamical systems. The approach is realized via differential inclusions following the Filippov theory. Three practical examples are considered.

1 Introduction

In the last years the main general methods for stabilization of chaos: slightly perturbations of a system parameter (the most known being the OGY method [1]) and changes in the system variables in the form of instantaneous pulses (method introduced by Güemez and Matías (see [2, 3]) have proved to be of a real interest. Thus the chaotic dynamical systems becomes an unlimited reservoir of stable behaviour. While the first algorithms work in the cases when we have access to some system parameter without changing the state variables, the second class of methods are useful in the cases when the system parameters are unaccessible, namely in the cases of certain chemical, biological electrical circuits etc.

The method considered in this paper performs changes in the system variables every time interval δ in the form

$$x(t) \leftarrow x(t)(1+\lambda) \tag{1}$$

where x are the state variable and the pulse λ can be positive or negative. This algorithm was applied to continuous[1] and discrete chaotic dynamical systems (see [2, 3]).

[1] The continuity/discontinuity is considered in this paper with respect to the state variables, the systems being considered time continuous.

We apply this algorithm to a class of piecewise-continuous dynamical systems which can be described by state discontinuous right-hand sides differential equations. Because the i.v.p. which models these systems may have not any solutions in the classical sense, the i.v.p. is transformed via Filippov regularization [4] into a differential inclusion which may have even several generalized (Filippov) solutions. These solutions can be numerical approximated using special numerical methods for differential inclusions. In this paper the forward Euler method was used.

The study of this class of piecewise continuous dynamical systems, called switch dynamical systems too, has been the subject of much ongoing research (see for instance [4]),.the past few years having seen a dramatic increase of interest in both the academic and industrial world. Thus, this class of dynamical systems can be found in many different branches of applied science and engineering. Examples include impacting machines, dry friction, impacts in mechanical devices, systems oscillating under the effect of an earthquake, power circuits, forced vibrations, switching in electronic circuits, elasto-plasticy, control synthesis of uncertain systems and many others (see e.g.[5, 6, 7, 8] and their references).

Therefore, like in the continuous case, the chaos control represents an important objective.

In order to apply the change of variables (1), some mathematical results on differential inclusions are used.

In this paper the control algorithm was applied to a class of known electronic circuits, but it works successfully to many other branches.

The paper is organized as follows: Section 1 presents the initial value problem which model the switch systems together with few underlying results; Section 2 treats the change of variables (1); Section 3 presents the applications to Chua, Sprott and Guanron circuits

2 Switch dynamical systems

The initial value problem (i.v.p.) which models switch dynamical systems is a Cauchy like problem

$$\dot{x}(t) = f\left(x(t)\right) + \sum_{i=1}^{n} \alpha_i \operatorname{sgn} x_i(t)\, e^i, \quad x(0) = x_0, \quad t \in I = [0, \infty)\,, \qquad (2)$$

where the vector-valued function $f:\mathbb{R}^n \rightarrow \mathbb{R}^n$ is considered to be continuous in $\mathbb{R}^n$, the sign function $sgn : \mathbb{R} \rightarrow \mathbb{R}$, is the known sign function, e^i are the canonical unit vectors ($e^1 = (1, 0, 0, ..., 0)$, $e^2 = (0, 1, 0, ..., 0)$,...) and α_i are real constants. The discontinuity is due to the sign functions. Thus, the continuity domain consists in a finite number of open regions $D_i \subset \mathbb{R}^n$, $i = 1, 2, ..., p$, the discontinuity set M (the set of points where the sign functions vanish) being given by $M = \mathbb{R}^n \backslash \bigcup_{i=1}^{p} D_i$.

For instance for the problem $\dot{x} = 2 - 3\,sgn(x)$, $D_1 = (-\infty, 0)$, $D_2 = (0, \infty)$, $M = \mathbb{R} \setminus (D_1 \bigcup D_2) = \{0\}$.

Because the i.v.p. (2) may have not any solutions in the classical sense, Filippov introduced the differential inclusions approach [4]. Thus he transformed the i.v.p. (2) into a multi-valued Cauchy one

$$\dot{x}(t) \in F(x(t)), \qquad x(0) = x_0, \quad for\ a.\ a^2.\ t \in I\,, \tag{3}$$

where instead of a differential equation, one obtain a differential inclusion. Hence $\dot{x}$ belongs to a set of values, $F(x)$, instead of a single value, $f\,(x)$, for some value of t .

$F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued function into the set of all subsets of $\mathbb{R}^n$ which can be obtained by the so called *Filippov regularization*

$$F(x) = co \lim_{x' \to x} f\,(x') \tag{4}$$

where co is the convex hull and $\lim_{x' \to x} f\,(x')$ is the set of all limits of all convergent sequences $f\,(x_k)$ with $x_k \to x$. For $x \in M$, $F(x)$ is a set, while for $x \notin M$, $F(x)$ consists in a single point $f\,(x)$. As example for the sgn function the Filippov regularization gives the following set-valued function

$$Sgn(x) = \begin{cases} -1 & for \quad x < 0, \\ [-1,\,1] & for \quad x = 0, \\ +1 & for \quad x > 0, \end{cases}$$

which for $x = 0$ is the whole segment $[-1,\,1]$.

Applying the Filippov regularization, the i.v.p. (2) becomes

$$\dot{x}(t) \in f\,(x(t)) + \sum_{i=1}^{n} \alpha_i \operatorname{Sgn} x_i(t)\, e^i, \quad x(0) = x_0, \quad for\ a.a.\ t \in I. \tag{5}$$

As example, the discontinuous i.v.p. $\dot{x} = 2 - 3sgn(x)$, $x(0) = 0$ becomes $\dot{x} \in 2 - 3\,Sgn(x)$, $x(0) = 0$ for a.a. $t \in [0, \infty)$.

Now, the i.v.p. (2) will be treated via (5) following the Filippov way. Hence as solution for (2) a generalized solution for (5) will be considered. While the i.v.p. (2) may have not any classical solutions, the i.v.p. (5) may have even several solutions, called Filippov or generalized solutions (a generalized solution of the i.v.p. (3) (or (5)) is an absolutely continuous vector-valued function $x\,(\bullet) : [0, \infty) \to \mathbb{R}^n$ verifying the i.v.p. (3) (or (5)) for a.a. $t \in [0, \infty)$). The background of differential inclusions and their solutions can be found in [9] and [10].

The conditions under which the i.v.p. (5) admits solutions and defines a dynamical systems are presented in [11] and will be not considered here.

Several properties of switch dynamical systems were analyzed by the author as: continuous approximation [12], synchronization [13] and anticontrol of chaos [14].

In order to obtain a numerical solution of (2) special numerical methods for differential inclusions can be used. The simplest one, which is used in the present paper, is the forward Euler method (see [15] for the background on numerical methods for differential methods).
Using the known notations, the explicit Euler method for differential inclusions gives a sequence (y_k) approximating the real trajectory

$$y_{k+1} = y_k + h\,\eta_k, \quad y_0 = x_0, \quad k = 1, 2, ...$$

where $\eta_k \in F(y_k)$ is chosen following some selection strategies (see [16] for selection strategies) and h is the step-size. Here, we chosen η_k randomly. For instance, for $F(x) = Sgn(x)$, $\eta_k \in F(y_k)$ means that η_k takes randomly a value between $[-1, 1]$.

3 The control algorithm

Let consider the following simplified form of (5)

$$\dot{x}(t) = g(x(t)), \quad x(0) = x_0, \tag{6}$$

with $g : \mathbb{R}^n \to \mathbb{R}^n$ a piecewise-continuous function. Then, using the transformation (1) to the switch system (6) one obtain the following controlled system

$$\dot{x}(t) = \begin{cases} g_1(x(t)) & \text{if } t \bmod \delta = 0 \\ \\ g(x(t)) & \text{if } t \bmod \delta \neq 0 \end{cases}, \tag{7}$$

where $g_1(x(t)) = g\left[(1+\lambda)\,x\right]$.

If g is homogenous we can introduce the following important result

Theorem 1 *Let the switch dynamical system (6) with g homogenous. Then the switch dynamical systems*

$$\begin{aligned} \dot{x}(t) &= g\,(x(t)), & \text{(8i)} \\ \dot{x}(t) &= (1+\lambda)\; g\,(x(t)), & \text{(8ii)} \end{aligned}$$

have the same trajectories.

Proof. In [4] (Theorem 6, p.105) is proved that if $p(x) > 0$ is a continuous function, then the equations $\dot{x}(t) = g(x(t))$ and $\dot{x}(t) = p(x(t))\,g(x(t))$ have the same trajectories. Next, from the homogeneity of g it follows that $g\left[(1+\lambda)\,x\right] = (1+\lambda)\; g(x)$. Choosing λ small enough for $t \bmod \delta = 0$ one obtain $p(x(t)) = p(t) = 1 + \lambda > 0$ and the theorem is proved. ∎

The relation (7) can be easily utilized to our i.v.p. (2).

Remark 2 *i) As in the continuous case, if* g *is nonhomogenous, the stabilized orbits could be generally only very closed, but not identical, to one of the stable orbits of the original system, while if* g *is homogenous the trajectory of any solution of (8 ii) is also the trajectory of some solution of (8 i).*
ii) At least in the homogenous case, following the Theorem 1, the same behaviour should be found to both systems (controlled and uncontrolled).
iii) The difference between the continuous and discontinuous case appears only in the discontinuity points $x \in M$*.*
iv) As in the continuous case, for each state variable, different values for λ *(even* 0*) could be used.*
v) The algorithm depends strongly on the characteristics of the used numerical method, especially the step-size. In the present paper we chosen the step-size which, for small changes on his size, dooes not affect the results.

4 Applications

We chosen for applications the case of three representative electronic circuits where the algorithm could practical verified too. These kind of circuits can be easily implemented by circuitry in the laboratory as in [17, 18, 19]. Generally the negative values for λ were useful, but few cases with $\lambda > 0$ were found. The biggest values for δ and the smaller values for λ were used in order to do not change semnificantly the system structure.

When a closed trajectory was stabilized, the same initial conditions (of the controlled and uncontrolled systems) were used.

For the sake of simplicity in the next we consider only the differential equations without the initial conditions.

All the images were obtained with a Turbo Pascal program which plots the projections in the phase portraits and time series. The numerical integration was realized for $t \in [t_0 = 0, t_{\max}]$. In the rigth colomn by $\delta = 0$ and $\lambda = 0$ one understain that the system is uncontrolled.

4.1 Chua circuit

In [20] is presented the following nonhomogenous generalized mathematical model of the classical Chua's circuit

$$\begin{aligned} \dot{x}_1 &= -2.57x_1 + 9\,x_2 + 3.86\; sgn\,(x_1), \\ \dot{x}_2 &= x_1 - x_2 + x_3, \\ \dot{x}_3 &= -\alpha\, x_2\,, \end{aligned} \tag{9}$$

For $\alpha = 17$ the behaviour is chaotic (Fig.1(a)) (see [12] too for some mathematical characteristics of this system). For for $\delta = 0.01$ and $\lambda = -0.002$ two controlled trajectories reach two stable fixed points (Fig.1(b)), which are different from those of the uncontrolled system (see Remark 2i).

Figure 1(a)(b) (Here or to the end of the paper)

4.2 Sprott circuit

One of the circuits introduced in [18] is modelled by the following discontinuous differential equations

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= x_3,\\
\dot{x}_3 &= -x_1 - x_2 - \alpha x_3 + sgn(x_1),
\end{aligned} \tag{10}$$

The chaotic behaviour, for $\alpha = 0.5$, can be seen in Fig.2(a)). Taking $\delta = 0.05$ and $\lambda = -0.005$ one stabilize the chaotic behaviour and two fixed point are reach (Fig.2(b)).

Figure 2(a)(b) (Here or to the end of the paper)

4.3 Chen circuit

A modified Chen system [21], which shares several important qualitative properties with Lorenz model, is modelled by the following homogenous discontinuous differential equations [22]

$$\begin{aligned}
\dot{x}_1 &= a\,(x_2 - x_1),\\
\dot{x}_2 &= sgn(x_1)(c - a - x_3) + c\,q\,x_2,\\
\dot{x}_3 &= sgn(x_2)\,x_1 - b\,x_3,
\end{aligned} \tag{11}$$

Choosing $a = 1.18$, $b = 0.168$, $c = 7$ and $q = 0.1$, like in [22], the system behaves chaotically (Fig.3(a)). For $\delta = 0.05$ and $\lambda = 0.002$ a periodic closed trajectory was obtained (Fig.3(b)). The same kind of motion was obtained but with a higher periodicity was obtained for $\delta = 0.025$ and $\lambda = -0.01$.

This system represents one of the few cases when positive value for λ. could used in order to stabilize the chaotic behaviour.

The right-hand side being a homogenous function the Theorem 1 can be applied.

Figure 3(a)(b)(c) (Here or to the end of the paper)

5 Conclusion

In this paper we applied in a class of switch dynamical systems modelled by (2) the algorithm of suppression of chaos through changes in the system variables introduced by Güémez and Matías in the class of continuous dynamical systems.

For the homogenous case the similarity between the set of the trajectories of the controlled and uncontrolled systems was proved. In the nonhomogenous case the most one can hope is to obtain stabilized trajectories closed to some stabilized trajectories of the uncontrolled system

We have applied the method to the stabilization of chaos in three practical examples which model electronic circuits. Using the algorithm we have stabilized periodic orbits and fixed points.

An open problem, as for the continuous case, is the choose of the step-size of the numerical method utilized in order to be sure that the results are correct. In this purpose the introduction of the shadowing technique[3] in the class of discontinuous dynamical systems -already studied for the continuous case (see e.g. [23])- would be of a real interest.

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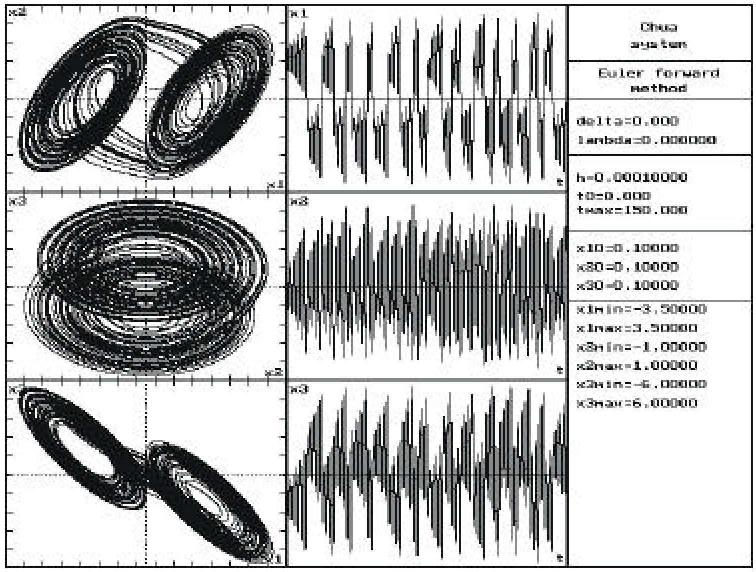
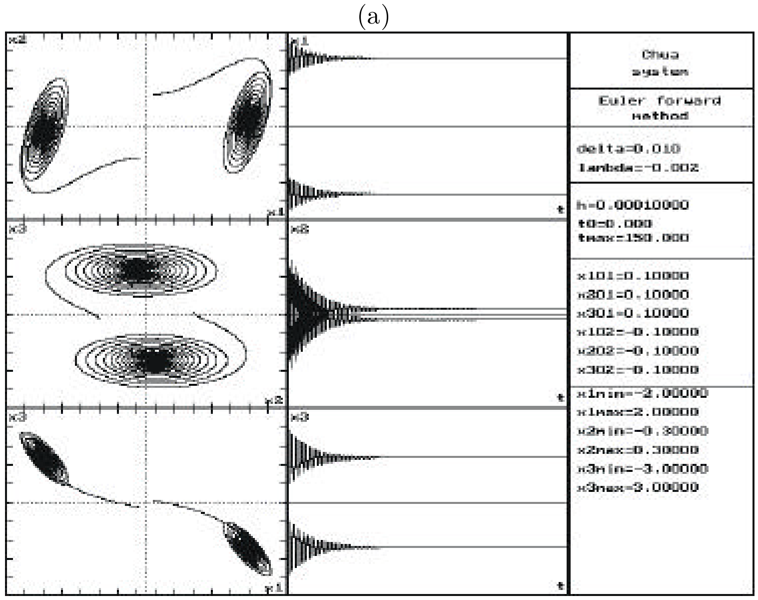
Figures

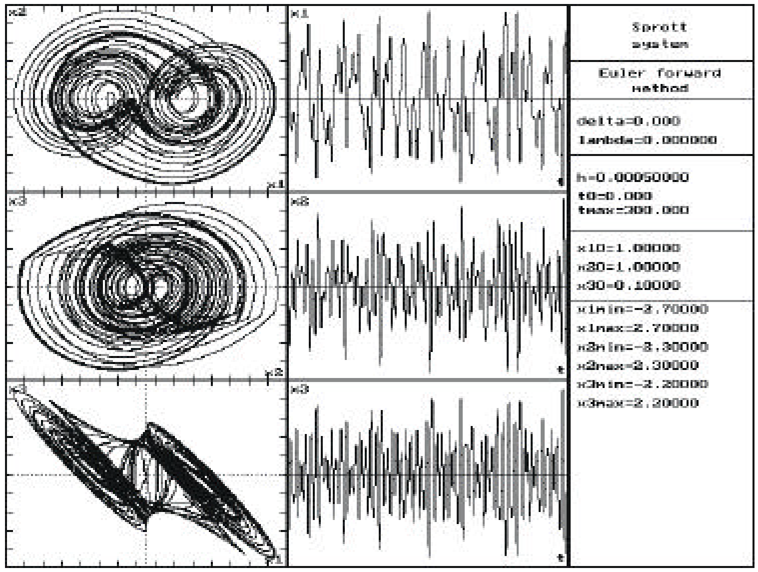


(a)



(b)

Fig.1(a)(b)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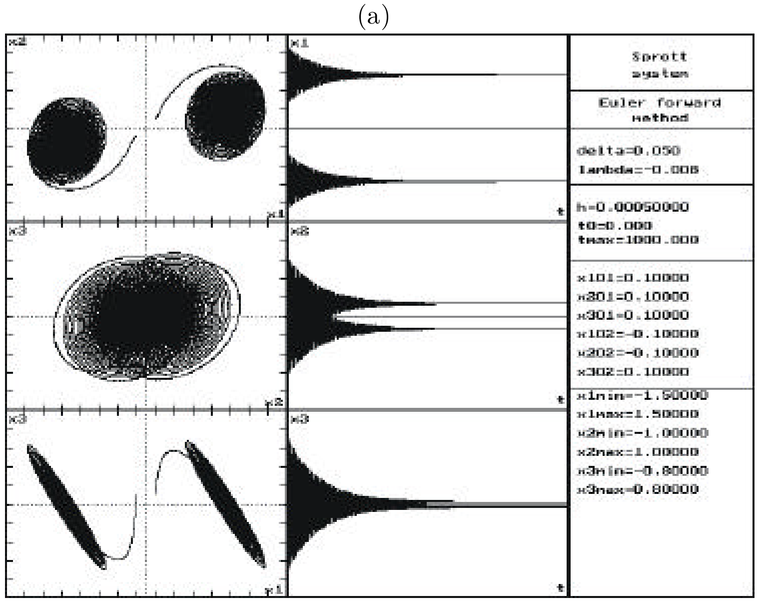
(a)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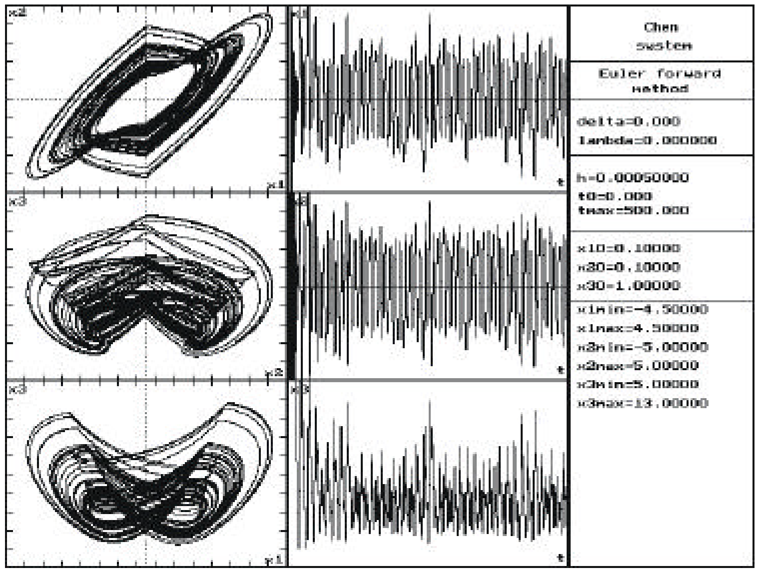
(b)

Fig.2(a)(b)


Chen
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Euler forward
method
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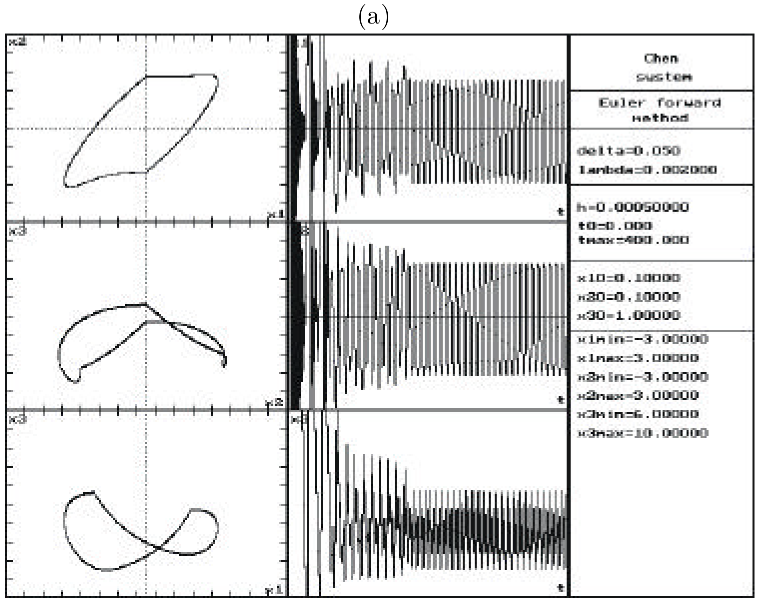
(a)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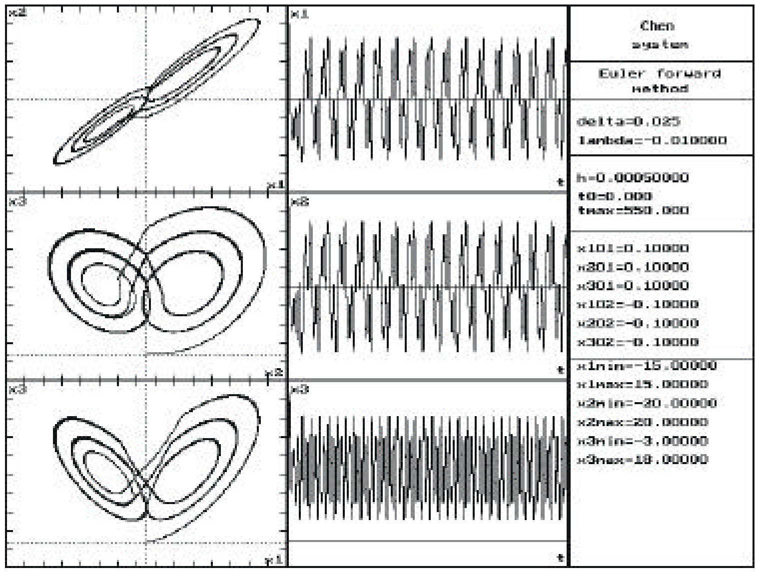
(b)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x2min=-20.00000
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x3min=-3.00000
x3max=18.00000

(c)

Fig.3(a)(b)(c)

Figure captions

Fig.1. (a) A chaotic trajectory of the Chua system (9). (b) two stable fixed points of the controlled system for $\delta = 0.01$ and $\lambda = -0.002$.

Fig.2. (a) A chaotic trajectory of the Sprott system (10). (b) two stable fixed points of the controlled system for $\delta = 0.05$ and $\lambda = -0.005$.

Fig.3. (a) A chaotic trajectory of the Chen system (11). Two stabilized periodic motions of the controlled system for (b) $\delta = 0.05$ and $\lambda = 0.002$ (the first steps were omitted in the phase portraits) and. (c) $\delta = 0.025$ and $\lambda = -0.01$.