# CHAOTIFYING DISCONTINUOUS DYNAMICAL SYSTEMS VIA TIME-DELAY FEEDBACK ALGORITHM

Shortened title:
CHAOTIFYING DISCONTINUOUS DYNAMICAL SYSTEMS

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**Abstract**

In this paper the anti-control technique of chaos to systems continuous with respect to the state variable using a time-delay feedback technique introduced by Wang, Chen and Yu, is adapted to a class of dynamical systems discontinuous with respect to the state variable. The considered discontinuous initial value problem is transformed into a differential inclusion using the Filippov regularization. Then, certain results on existence and uniqueness of solutions to differential inclusions are used to define our class of discontinuous dynamical systems. Afterwards, introducing an adequate concept of derivative for the considered discontinuous functions, we show that the algorithm for continuous dynamical systems can be adapted to our class of discontinuous problems. Three examples are given.

*Keywords:* Filippov regularization; differential inclusions; switch dynamical systems; generalized derivative; chaotification.

# 1 Introduction

Chaos can become very useful under circumstances. For example, it is important in the biological systems as human brain [Freeman, 1995], heartbeat regulation [Brandt and Chen, 1997], liquid mixing [Ottino et al., 1992], resonance prevention in mechanical systems [Georgiu and Schwartz, 1999], secure communications [Hasler and Schimming, 2000] etc.

Therefore, besides the control of chaos (see the pioneering work of Ott et al. [1990]), a natural, yet non-trivial question, is the anti-control i.e. whether one can make a given system chaotic or enhance the existing chaos of a chaotic system by using small control (see e.g. [Chen and Dong, 1993, Chen and Dong, 1998, Kastner-Maresch and Lempio, 1993, Lakshmanan and Murali, 1996, Shinbrot et al., 1993]). Several mathematically rigorous anti-control algorithms for discrete and continuous dynamical systems, were developed by Chen and collaborators (see e.g. [Chen and Lai, 1996] and [Chen and Lai, 1998]). Note that the continuity concept could refers both the time and state variable. Here, by dis/continuous dynamical systems we mean systems dis/continuous with respect to the state variable.

Differential equations with discontinuities with respect to the state variable modeling dynamical systems (d.s) occur in many real problems and are widely used as simplified mathematical models of physical systems although the initial value problems (i.v.p.) need not have any classical solutions. Hence, sometimes physical laws are expressed by discontinuous functions, for example, a discontinuous dependence of the friction force on the velocity in the cases of dry friction, brake processes with locking phase, oscillating systems with combined dry and viscous damping, elasto-plasticity, electrical circuits, forced vibrations, convex optimization, control synthesis of uncertain systems etc. (see e.g. [Popov, 1962, Popp and Stelter, 1990, Wiercigroch and de Kraker, 2000] and the references therein).

The discontinuous functions we consider in this paper are piece-wise continuous (see [Filippov, 1988]), i.e. functions continuous on a finite number of open domains $D_i \subset \mathbb{R}^n$, $i = 1, ..., p$ in each of which the functions being continuous up to the boundary, and having finite (possible different) limits from different boundary points (bounded discontinuities). The set of zero measure $M = \mathbb{R}^n \setminus \cup_{i=1}^{p} D_i$ contains the boundaries of $D_i$ and represents the set of discontinuity points of $f$. This class of i.v.p. can be modeled by the following autonomous i.v.p.

$$\begin{aligned} \dot{x}(t) &= f(x(t)) := g(x(t)) + \sum_{i=1}^{n} \alpha_i \operatorname{sgn} x_i(t)\, e^i, \\ x(0) &= x_0, \quad t \in I = [0, \infty), \end{aligned} \tag{1}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function continuous with respect to the state variable, $\alpha_i \in \mathbb{R}$, and $e^i$ denote the $i$-th canonical unit vectors in $\mathbb{R}^n$.

Since the system is autonomous, we can assume, without loss of generality, that the initial condition is given at $t = 0$. The restriction to autonomous problems is not restrictive: one can introduce a new variable $x_{n+1}$ satisfying $\dot{x}_{n+1} = 1$ and $x_{n+1}(t_0) = t_0$.

Due to the right-hand side discontinuity, the i.v.p. (1) need not have any solutions and another concept of solution must be used.

**Example.** Consider the suggestive example of discontinuous right-hand side equation [Filippov, 1988]

$$\dot{x} = 1 - 2\, sgn\,(x),$$

which has, for $x \neq 0$, the classical solutions

$$x(t) = \begin{cases} 3\,t + C_1, & x < 0 \\ -t + C_2 & x > 0 \end{cases}, C_1, C_2 \in \mathbb{R}\,.$$

As $t$ increases, these solutions tend to the line $x = 0$, but it cannot be continued along this line (the function $x(t) = 0$ does not satisfy the equation in the usual sense).

Our first goal is to find the assumptions on $g$ on which the i.v.p. (1) defines a d.s. For this purpose, a definition of d.s., using the existence and optionally the uniqueness, will be used. Due to the possible lack of solutions, the Cauchy problem (1) is transformed into a differential inclusion (d.i.) using the well known Filippov regularization [Filippov, 1988]. Enjoying enough regularity, the obtained d.i. may have several generalized solutions.

The mathematical background of the i.v.p. (1) can be found in [Danca, 2002a,b and c]. In [Danca and Codreanu, 2002] the i.v.p. was treated using a continuous approximation of the discontinuity.

Next, a generalized concept of derivative for our class of functions, is introduced and the anti-control technique of chaos for continuous systems, proposed by Wang, Chen and Yu [Wang et al., 2000], which uses the modern geometric theory of nonlinear control (see e.g. [Isidori, 1995] or [Sastry, 1999]), is adapted to our class of i.v.p. (1).

The paper is organized as follows. Section 2 treats the assumptions under which the i.v.p. (1) models a switch d.s. For this purpose the existence of solution to the i.v.p. (1), the Filippov regularization used to transform the i.v.p. into a d.i., and the explicit Euler method for d.i. is presented. Sec. 3 presents a special derivative for the right-hand side of the i.v.p. (1) and the anti-control technique and in Sec. 4 three applications are presented.

# 2 Switch dynamical systems

Next, a few notions necessary to define our class of discontinuous d.s. are presented (details can be found in [Danca, 2002a and c]).

**Definition 2.1.** A single-valued function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies a *growth condition* on $\mathbf{R}^n$ if there exist constants $K_1,\, K_2 \geq 0$ with

$$\|f\,(x)\| \leq K_1\,\|x\| + K_2,$$

for all $x \in \mathbb{R}^n$.

All the practical examples found by us can be obviously verified to satisfy the growth condition.
Because of the lack of the solutions to i.v.p. (1) we restart the i.v.p. into a set-valued one

$$\dot{x} \in F(x),\ x(0) = x_0,\ \text{for almost all } t \in I\,, \tag{2}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued function which can be defined in several ways. For our class of functions $f$ , defined in (1), the simplest convex definition of $F$ is obtained by the so-called *Filippov regularization* [Filippov, 1988]

$$F(x) = \bigcap_{\varepsilon>0}\ \bigcap_{\mu\,(M)=0} \overline{conv}\, f\,((x + \varepsilon\, B)\backslash\, M)\,, \tag{3}$$

where $B$ is the unit ball in $\mathbb{R}^n$, $\mu$ is the Lebesgue measure and $\overline{conv}$ is the closed convex hull. In the points where the function $f$ is continuous, $F(x)$ consists of one point which coincides to the value of $f$ at this point. In the discontinuity points, the set $F(x)$ is given by (3).
As an example the Filippov regularization applied to *sign* function gives us the set-valued function

$$Sgn\,(x) = \begin{cases} \{-1\} & x < 0 \\ [-1,\, 1] & x = 0 \\ \{+1\} & x > 0 \end{cases},$$

Applying the Filippov regularization to the i.v.p. (1) this becomes

$$
\begin{aligned}
\dot{x} &\in F(x) = g(x) + \sum_{i=1}^{n} \alpha_i Sgn\,(x_i)\, e^i, \\
x(0) &= x_0, \text{ for almost all } \ t \in I.
\end{aligned} \tag{4}
$$

**Definition 2.2.** A *generalized* (Filippov) solution to i.v.p. (1) is an absolute continuous function $x : I \to \mathbb{R}^n$ satisfying the differential inclusion (4) almost everywhere in $I$ .

The existence theorem for differential inclusions is a Péano theorem and can be found in many works and various forms (see e.g. [Aubin and Cellina, 1984, Aubin and Frankowska, 1990, Kastner-Maresch and Lempio, 1993]. The proof for the general case of a differential inclusion can be found in e.g. [Aubin and Cellina, 1984] and for the i.v.p. (4) in [Danca, 2002a or Danca 2002c].

**Example**. Let consider the discontinuous i.v.p. $\dot{x} = sgn\,(x)$, $x(t_0) = 0$. There is no classical solution starting from 0. However, considering the corresponding set-valued i.v.p. $\dot{x} \in F(x) = Sgn\,(x)$, $x(t_0) = 0$, there are multiple Filippov solutions: $x(t) = 0$ for $t \leq t_0$ and $x(t) = \pm(t - t_0)$ for $t > t_0$, where $t_0 \geq 0$.

*Remark 2.1.* In [Danca, 2002a,c] it is proved that the uniqueness of the solution to the general i.v.p. (4) is verified if $g$ is Lipschitz continuous and all the coefficients $\alpha$ are negative.
For our class of functions $f$ defined in (1) , the positiveness of some $\alpha_k$ seems to be adequate for non-uniqueness . In [Filippov, 1988, p. 50] there are presented geometrical proofs to study the uniqueness.

**Example** The set-valued i.v.p. $\dot{x} \in -Sgn\,(x)$, $x(0) = x_0$, has a unique generalized solution . Hence, for $x_0 > 0$, the solution is $x(t) = x_0 - t$, for $t < x_0$ and $x(t) = 0$, for $t \geq x_0$ and the corresponding trajectory, starting from $x = 0$ , can be continuously extended for $t \geq x_0$. If $x_0 < 0$ then $x(t) = -x_0 + t$ for $t < x_0$ and $x(t) = 0$ for $t \geq x_0$.

Using the above concepts and results, we can introduce the following definition which states the conditions under which the i.v.p. (1) defines a discontinuous switch d.s.

**Definition 2.3.** The i.v.p. (1) is said to define a *generalized switching d.s.* on $\mathbb{R}^n$ if for every $x_0 \in \mathbb{R}^n$ there exists a solution of the i.v.p. defined for almost all $t \in I$. If the solution is almost everywhere unique, then the i.v.p. is said to define a *switch d.s.*

In [Danca, 2002a,c] it is proved that if $g$ is continuous and verifies a growth condition then the i.v.p. defines a generalized switch d.s. and if supplementary is Lipschitz continuous and all the coefficients $\alpha$ are nonpositive then the i.v.p. defines a switch d.s.

In [Danca and Codreanu, 2002] the class of i.v.p. (1) was treated using the approximate selection Theorem [Aubin and Cellina, 1984, Aubin and Frankowska, 1990].

In order to simulate the dynamics of the switch d.s. and to apply the anti-control algorithm, a numerical method for d.i. is necessary. Difference methods for d.i. are presented in many references (see e.g. [Dontchev and Lempio, 1992, Lempio, 1995, Kastner-Maresch and Lempio, 1993]). Here, we consider the classical explicit Euler method.

Consider the i.v.p. (2). Let $N$ be a natural number $N \in \mathbb{N}' \subset \mathbb{N}$, $\mathbb{N}'$denoting a subsequence of $\mathbb{N}$ tending to infinity, the integration step-size $h = (T - t_0)/N$ and an equidistant grid

$$t_0 < t_1 < ... < t_N = T.$$

We associate to (2) a sequence of discrete-time inclusions in the form

$$\begin{gathered} y_{k+1} \in G_k^N(h;\, y_k)\,, \\ k = 0, 1, ..., N-1,\ y_0 = x_0, \end{gathered} \tag{5}$$

where $G_k^N : [t_0, T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are discrete-time set-valued maps. A solution of (5), for a given step-size $h$, is any sequence of $N+1$ vectors $y_0, y_1, ..., y_N$, satisfying (5) for $k = 0, 1, ..., N-1$. The main problem is to define a family of mappings $G_k^N$ such that the solutions of the problem (5) approximate in some sense the solutions of the original problem (2). For the explicit Euler method the set-valued map $G_k^N$ is

$$G_k^N(h;\, y_k) = y_k + h\, F(t_k,\, y_k)\,. \tag{6}$$

The convergence theorem for the general case of i.v.p. (2) can be found e.g. in [Filippov, 1988, Theorem 1, pp.77], [Aubin and Cellina, 1984, Lemma 1, pp. 99], [Aubin and Frankowska, 1990, Theorem 10.1.3, pp. 390], or in the paper Lempio, 1995], and for our i.v.p. (4) in [Danca, 2002c].

Since in general the solution of the inclusion (5) is not unique, the main problem is to reasonably choose $y_{k+1}$ of $G_k^N(h;\, y_k)$ at each step of the discrete system. Therefore $y_{k+1}$ could be selected randomly, as in our numerical examples (see [Dontchev and Lempio, 1992] and [Kastner-Maresch and Lempio, 1993] for selection strategies). If the solution is unique, the whole sequence of approximations converges to this solution.

The maximal existence interval $[t_0, \infty)$ is obtained for the existence of both generalized and numerical solutions of (1).

All drawings in this paper were obtained with forward Euler method (5)-(6), using a Turbo Pascal code.

The considered class of discontinuous functions are ideally suited for electronic implementations because they can be accurately represented by resistors, capacitors, diodes and operational amplifiers (see e.g. [Sprott 2000]).

**Example 2.1.** Let us consider the following switch problem which models a generalization of the Chua circuit [Brown, 1993], [Yalcin et al. 2002] analyzed in [Danca 2002b]

$$\begin{array}{l} \dot{x}_1 = -2.57\,x_1 + 9\,x_2 + 3.86\,sgn\,(x_1) \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -\,\beta\,x_2 \end{array}, \tag{7}$$

$\beta > 0$ being the control parameter. Here, the set of discontinuity points belongs to the surface given by the equation $x_1 = 0$. The i.v.p. has no global classical solution on $[0, \infty)$ . The Filippov regularization gives us the following d.i.

$$\begin{array}{l} \dot{x}_1 \in -2.57\,x_1 + 9\,x_2 + 3.86\,Sgn\,(x_1) \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -\,\beta\,x_2 \end{array}, \tag{8}$$

with the corresponding set-valued function

$$F(x) = \begin{pmatrix} -2.57\,x_1 + 9\,x_2 \\ x_1 - x_2 + x_3 \\ -\,\beta\,x_2 \end{pmatrix} + 3.86\,Sgn\,(x_1)\,e^1$$

The i.v.p. defines a generalized switch d.s. because the function $g(x) = (-2.57\,x_1 + 9\,x_2,\ x_1 - x_2 + x_3,\ -\,\beta\,x_2\,)^T$ is linear and the solution is not unique (the nonuniqueness is proved in [Danca 2002c]). A chaotic trajectory, for $\beta = 15.7$ , (phase portraits and time series) is presented in Figs. 1 a, b. The chaotic behavior can be observed too from the Poncaré section with the plane $x_2 = 0.1$ (Fig.1 c), and in the bifurcation diagram of the phase variable $x_3$ versus the control parameter $\beta$ (Fig.1 d). In [Danca, 2002a] a Simulink (Matlab) scheme, which enables to obtain several informations on the system, is presented.

**Remark.** $i$) Chaos in uncontrolled switch d.s. will be understood in one of the classical senses, e.g. positive Lyapunov exponents, eventually with supplementary tools as bifurcation diagram (see e.g. [Brown and Chua, 1996]), while the chaos induced by the anti-control algorithm, will be understood in the sense of Li-Yorke [Li and Yorke, 1975] ( "period three implies chaos"), or Marotto [Marotto, 1978] for higher dimensional state spaces. In [Wang et al., 2000] an asymptotically approximate relationship between the time-delay equations (which appears in the anti-control algorithm), and difference equations is established.
$ii$) It can be easily proved that the equilibrium points of switch dynamical systems does not belong to the discontinuity surfaces.

**Example 2.2**. Let consider the following system which is a simplified model of the regulation systems of a steam turbine [Belea, 1983]

$$\begin{array}{l} \dot{x}_1 = a\ (x_3 - x_1 - \ sgn\,(x_1)) \\ \dot{x}_2 = x_1 - x_2 \\ \dot{x}_3 = -\ x_2,\ a > 0 \end{array} . \tag{9}$$

Again $g$ is continuous and verifies a growth condition. The system is a switch d.s. the solution of i.v.p. being unique (see Remark 2.1). The periodic behavior (hysteresis like motion for $a > 0$), is plotted in Fig.2.

Studying practical examples, we have observed that generally the uniqueness of the solutions in the cases of switch d.s. seems to imply a strong stability to the system for a wide range of the control parameters. In this meaning, chaos, if it does exist, is very frail.

**Example 2.3.** The following problem is a discontinuous variant of the chaotic d.s. presented in [Aziz-Alaoui and Chen, 2002] and models a generalized switch d.s.

$$\begin{array}{l} \dot{x}_1 = a\ (x_2 - x_1) - 0.5\,sgn\,(x_1) \\ \dot{x}_2 = x_1\,(c - a - x_3) + c\,d\,x_2 + 0.5\,sgn\,(x_2) \\ \dot{x}_3 = -x_2 x_1 - b\,x_3 + 3\,sgn(x_1) \end{array} . \tag{10}$$

A chaotic trajectory, obtained for $a = 1.18$, $b = 0.168$, $d = 0.1$ and the control parameter $c = 7$, is plotted in Figs. 3 a,b. The Poincaré section (with the plane $x_3 = 10$) and the bifurcation scenario plotted in Figs. 3 c,d show the chaotic behavior.

## 3 Anti-control of chaos

Since the classical derivative cannot be used to our class of functions, a new concept will be introduced (Danca, 2002b]).

Let $D_i$ be open subsets of $\mathbb{R}^n$, for $i = 1, 2, ..., p$, such that $\mathbb{R}^n = \overline{\bigcup_{i=1}^{p} D_i}$ and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a real single-valued function.

**Definition 3.1.** Let $f$ be differentiable on $\bigcup_{i=1}^{p} D_i$. We say that $f$ is *generalized differentiable* at $x^* \in \mathbb{R}^n$ if the following limits exists and is finite

$$\mathcal{D} f(x^*) := \lim_{x \to x^*} f'(x), \ x \in \bigcup_{i=1}^{p} D_i. \tag{11}$$

$\mathcal{D} f(x^*)$ will be called the *generalized derivative of* $f$ at $x^*$. We say that $f$ is *generalized differentiable* on $\mathbb{R}^n$ if it is so at every $x^* \in \mathbb{R}^n$.
A higher order generalized derivative at $x^* \in \mathbb{R}^n$ can be easily recursively defined

$$\begin{array}{c} \mathcal{D}^{(m)} f(x^*) = \mathcal{D}(\mathcal{D}^{(m-1)} f(x^*)) = \\ \lim\limits_{x \to x^*} f^{(m-1)}(x), \ x \in \bigcup\limits_{i=1}^{p} D_i \end{array}, \tag{12}$$

if $f$ is differentiable of order $m-1$, $m \geq 1$ on $\bigcup_{i=1}^{p} D_i$, and the limit (12) exists and is finite.

**Notation 3.1.** We denote the class of functions $f$ having generalized derivatives of order $m$ on $\mathbb{R}^n$ by $\mathcal{C}^m$.

**Example** Let us consider the function

$$f(x) = x - 2\,sgn\,(x)\,.$$

Here $D_1 = (-\infty\,, 0)$ and $D_2 = (0, \infty)$. In the discontinuity point $x^* = 0$, we have $D f(0) = \lim_{x \to 0} f'(x) = 1$ ( Fig.4).

It is easy to check the following proposition.

**Proposition 3.1.** *Let consider the i.v.p. (1) with* $g \in C^m[\mathbb{R}^n]\,, m \geq 1$ . *Then* $f \in \mathcal{C}^m$ *and*

$$\mathcal{D}^{(m)} f\left(x^*\right) = g^{(m)}(x^*),\ x^* \in \mathbb{R}^n.$$

In [Danca, 2002b] the class of switch d.s. (1) with $f \in \mathcal{C}^m$ was used to synchronize switch d.s.

In order to adapt the anti-control algorithm for continuous d.s. presented in [Wang et al., 2000] to switch d.s. let us consider first the affine and autonomous d.s. with no output

$$\dot{x} = f\left(x\right) + h(x)\, u, \tag{13}$$

where $f$ is defined in (1) $g$ and $h$ are smooth functions on $\mathbb{R}^n$ and $u \in \mathbb{R}$ is either a system parameter perturbation or an exogenous control input. Next, *smooth* will mean an large enough times (generalized) differentiable function. We want now to find if it is possible to determine a smooth nonlinear function $l$ (system output) satisfying $l\left(0\right) = 0$, such that the following nonlinear autonomous SISO (Single-Input Single-Output) d.s.

$$\begin{aligned} \dot{x} &= f\left(x\right) + h\left(x\right) u \\ y &= l\left(x\right), \end{aligned} \tag{14}$$

drives chaotically (even if the uncontrolled system is nonchaotic).
Let $x^*$ be an asymptotically stable equilibrium point of the uncontrolled system $(u \equiv 0)$. In order to design a feedback controller, $u(t)$, such that the behavior of the controlled system (13) becomes chaotic within a neighborhood of $x^*$, the differential geometric control theory ([Isidori, 1995, Sastry, 1999]) is applied to the d.s. (14).

A system with time-delay is inherently infinite dimensional, and is able to have complicated behavior such as bifurcations and chaos.
One of the simplest choice of $u$ (which clearly is not unique) for which the system becomes chaotic in the rigorous mathematical sense of Li and Yorke [Li and Yorke, 1975] would be

$$u(t) = \varepsilon \sin\left(\sigma\, l\left(x(t-\tau)\right)\right). \tag{15}$$

where $\varepsilon$ is the maximum amplitude of the control input $u$, $(\left| u\left(t\right)\right| \leq \varepsilon,\ \forall\, t \geq 0)$, $\tau > 0$ is a time-delay and $\sigma$ is a positive control parameter.
In order to find $l$ we need first some auxiliary results.
The *Lie derivative* of the smooth function $l$, with respect to a function $f$, is defined recursively

$$\begin{aligned} &L_f\, l\left(x\right) = \tfrac{\partial}{\partial x} l\left(x\right)^T f\left(x\right) \\ &L_f^i\, l\left(x\right) = L_f\, \left(L_f^{i-1}\, l\left(x\right)\right) = \\ &\left[\tfrac{\partial}{\partial x}\left(L_f^{i-1}\, l\left(x\right)\right)\right]^T f\left(x\right),\ i > 1. \end{aligned}$$

The *Lie bracket* of two smooth functions $h_1$ and $h_2$, is defined recursively as follows

$$\begin{array}{l} ad_{h_1}h_2(x) = \frac{\partial}{\partial x}h_2(x)h_1(x) - \frac{\partial}{\partial x}h_1(x)h_2(x) \\ ad^i_{h_1}h_2(x) = ad_{h_1}\left(ad^{i-1}_{h_1}h_2(x)\right) = \\ \frac{\partial}{\partial x}\left(ad^{i-1}_{h_1}h_2(x)\right)h_1(x) - \frac{\partial}{\partial x}h_1(x)\, ad^{i-1}_{h_1}h_2(x),\ i > 1. \end{array}$$

**Definition 3.2.** Let $f_1, f_2, ..., f_m$ smooth functions. The linear span of these functions

$$\Delta := span\left\{f_1(x),\ f_2(x), ..., f_m(x)\right\},$$

defined on an open subset of $\mathbb{R}^n$, is said to be *involutive* if for $\tau_1(x), \tau_2(x) \in \Delta$ we have $ad_{\tau_1}\tau_2(x) \in \Delta$.

**Definition 3.3.** The d.s. (14) is said to have a *relative degree* $r$ at $x^*$ if there exists a neighborhood $D$ of $x^*$ such that

$$\begin{array}{l} (i)\ L_hL^k_f\, l\,(x) = 0,\ 0 \leq k < r-1; \\ (ii)\ L_hL^{r-1}_f\, l\,(x) \neq 0,\ \text{for all}\ \ x \in D\,. \end{array}$$

The main mathematical tool used to find $u$ is the following lemma

**Lemma 3.1.** ([Isidori, 1995, Sastry, 1999]) *The d.s.* (14) *has relative degree* $n$ *at* $x^*$ *if and only if there exists a neighborhood* $D$ *of* $x^*$ *such that*

$$\begin{array}{l} (i)\ rank\ \left(h(x),\ ad_f\, h(x), ..., ad^{n-1}_f h(x)\right) = n \\ \text{for all}\ x \in D; \\ (ii)\ span\left\{h(x),\ ad_f\, h(x), ..., ad^{n-2}_f h(x)\right\} \\ \text{is involutive in}\ D\,. \end{array}$$

*In this case, the system output* $y = l\,(x)$ *is a solution of the set of* $n-1$ *first order linear partial differential equations*

$$\frac{\partial}{\partial x}l\,(x)\left[h\,(x),\ ad_f\, h\,(x), ..., ad^{n-2}_f h\,(x)\right] = 0\,. \tag{16}$$

Let now consider the i.v.p. (1). Then we have the following theorem

**Theorem 3.1.** *Let the i.v.p. (1) with* $g \in C^m\left[\mathbb{R}^n\right]$ *verifying the growth condition and* $x^*$ *be an asymptotically stable equilibrium point of the non-controlled system. Then the switch d.s. can be drive chaotically if and only if the system has the relative degree* $n$ *.*

*Proof.* Using the generalized derivative we obtain $\frac{\partial f(x)}{\partial x} = \frac{\partial g(x)}{\partial x}$ and Lemma 3.1 can be used. Hence, if the conditions $i$ and $ii$ hold, the small-amplitude time-delay feedback (15) can be used to create or enhance the chaos. ■

*Remark 3.1.* $i$) There are cases where the neighborhood $D$ can be chosen so that it contains several equilibrium points, which in the case of switch d.s. are separated by the discontinuity surfaces. Then, under the anticontrol algorithm, the underlying separated attractors could merge, for certain values of $\varepsilon$, $\sigma$ and $\tau$, into a scroll chaotic attractor.
$ii$) If the (generalized ) switch d.s. contains in a neighborhood $D$ only a single stable equilibrium point, then the d.s. is in fact a smooth one in this neighborhood and the classical derivative can be used. If in $D$ there are several stable equilibrium points separated by the discontinuity surfaces, then the generalized derivative near these surfaces is necessary.
$iii$) The conditions $i$ and $ii$ in Lemma 3.1 are the necessary and sufficient conditions for solutions of (16). While the condition $i$ is more laborious than $ii$, the last one is more subtle and we found several cases when this condition does not holds.

# 4 Applications

**4.1.** Let us consider the switch Chua circuit (7) which has the equilibrium points $X^*_{1,2}(\pm 1.5,\, 0,\, \mp 1.5)$. For $\beta = 23.5$ the trajectory tends to one of the equilibrium points (Figs.1d and 5a). If we denote $\beta(t) = \beta + \delta\,\beta(t)$, (with $u = \delta\,\beta$), the controlled system becomes

$$\dot{x} = \begin{pmatrix} -2.57x_1 + 9x_2 \\ x_1 - x_2 + x_3 \\ -\,\beta\, x_2 \end{pmatrix} + 3.86\, sgn\,(x_1)\, e^1 + \\ h\,(x)\,\delta\,\beta\,(t)\,.$$

with $h(t) = (0,\, 0,\, -\, x_2)^T$. The first two Lie brackets are

$$ad_f\, h(x) = \begin{pmatrix} 0 \\ x_2 \\ -x_1 + x_2 - x_3 \end{pmatrix},$$
$$ad_f^2 h(x) = \\ \begin{pmatrix} -\,9\, x_2 \\ -x_2 + x_1 + x_3 \\ 3.57\, x_1 + (2\,\beta - 10)\, x_2 - 3.86\, sgn\,(x_1)\, + x_3 \end{pmatrix}.$$

Let denote the matrix $A = \big(h(x),\, ad_f\, h(x),\, ad_f^2 h(x)\big)$. The rank is three because $|A| = 9\, x_2^3 \neq 0$ for all $x \in U$, $x_2 \neq 0$ where $U$ is a neighborhood of one of the fixed points $X^*_{1,2}$. Hence the assumption $i$ is verified. In order to verify $ii$ we have $ad_h\,(ad_f\,h(x)) = (0,\,0,\,-2\,x_2)^T = -2\,h(x)\,,$ i.e. $span\,(h(x),\,ad_f\,h(x))$ is involutive for $x \in U,\, x_2 \neq 0$. Therefore the system has relative degree three at the equilibrium points and $l$ can be obtained from the following system of partial derivatives

$$\begin{array}{l} \frac{\partial}{\partial x} l\,(x)\,h(x) = -\frac{\partial\,l\,(x)}{\partial\,x_3} x_2 = 0 \\ \frac{\partial}{\partial x} l\,(x)\,ad_f\,h\,(x) = 0 \\ \frac{\partial\,l\,(x)}{\partial\,x_2} x_2 + (-x_1 + x_2 - x_3)\,\frac{\partial\,l\,(x)}{\partial\,x_3} = 0 \end{array}\,,$$

with a solution

$$y(t) = l\,(x_1(t))\,.$$

Hence, we can take

$$\delta\,\beta\,(t) = \varepsilon\,\sin\,(\sigma\,(x_1(t-\tau)))\;.$$

If we chose $\varepsilon = 0.6$ and $\sigma = 40$ for the time delay $\tau = 1,$ the controlled system reach two separated chaotic attractors; each is near one of the two originally stable fixed points (Fig.5 b). The obtained chaotic motion can be deduced too from the Poincaré section with the plane $x_2 = 0.1$ (Fig.5 c) and bifurcation diagram (Fig.5 d). If we chose $\varepsilon = 1.5$ and $\sigma = 50,$ then for $\tau = 1,$ the two separated attractors merge into one double scroll chaotic attractor as it is visualized in Fig.5 e (see Remark 3.1 $i$).

**4.2.** For $c = 11$ the system (10) has a periodic motion (Fig.6a). The condition $i$ hold because $rank\,\big(h(x),\,ad_j h(x),\,ad_j^2 h(x)\big) = 3,$ but the condition $ii$ does not hold. Therefore the system has not the relative degree three. Nevertheless, choosing $h(x) = (0,\,d\,x_2 + x_1,\,0)^T$ and $l$ given by (16), $l\,(x) = x_1^2/2 - a\,x_3,$ with $a = 1.18,$ the system can be driven chaotically for $\varepsilon = 2$ and $\sigma = 4$ (see Figs.6 b, c and d where the phase portraits, time series, Poincaré section with the plane $x_3 = 10,$ and bifurcation diagram were plotted). This example shows the strength of chaotification induced by the time-delay control (15). However, when the Lemma 3.1 does not hold, there are cases when the chaos cannot be induced, as in the following example.

**4.3.** Let us consider the system (9). Again the condition $i$ is easy verified, while the condition $ii$ is not checked. In this case, despite our numerical experiments, the only we found were large size chaotic attractors in the phase space (of $10^5 \div 10^6$ order size) without practical interest . Actually, the attractors are not typical, but some bursts like (see Fig.7 where, the controller was chosen to be $y(x) = 15 \sin(35\, x_2(t-1))$ ). The difficult chaotification could be here explained by the robustness (structural stability) logically necessary to this system. Note that the apparent "regular" motion (the horizontal regions in the time representations) contains in fact many corners, typically for explicit numerical methods for d.i. These probably correspond to "grazing regions", i.e. regions in phase planes corresponding to the discontinuity surfaces, where the trajectories arrive tangentially. These corners can be reduced by using numerical methods with high order of consistency.

# 5 Conclusions

In this paper we presented a class of discontinuous d.s. using some results on existence and uniqueness of solutions to d.i. Here the d.i. are obtained with the Filippov regularization. Introducing a special derivative for the class of problems (1) the anti-control technique of continuous d.s. is applied to our switch d.s.

It seems that there are practical cases where the anti-control algorithm can be generally applied even if the system does not have the relative degree equal to the space phase dimension.

It would be of a real interest to find out if there are dynamical systems (continuous or not) which verifies Lemma 3.1 proving a strong stability. This could be another criteria for stability degree.

Note that if the minimum value of $\varepsilon$, for which the chaos appears, exceeds some value, the anticontrol would be not interesting since the underlying mathematical model changes significantly.

The choose of the time-delay $\tau$ has not a significant influence in the cases we studied.

The generalized derivative for the class of i.v.p. (1) was proved to be an ideal tool to introduce other applications of the classical chaos theory for continuous d.s. to switch d.s. (e.g. the OGY control for chaotic switch d.s., which will be a subject for a future paper).

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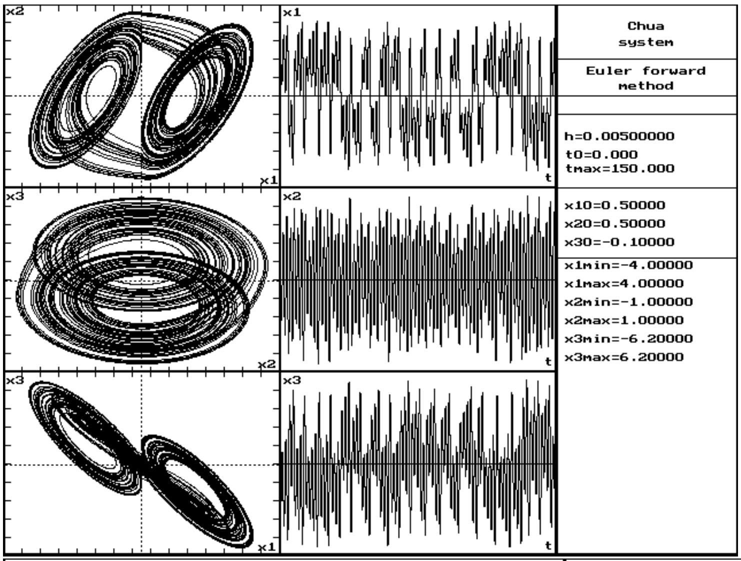
x2
x1
x1
t
x3
x2
x2
t
x3
x1
x3
t
Chua
system
Euler forward
method
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t0=0.000
tmax=150.000
x10=0.50000
x20=0.50000
x30=-0.10000
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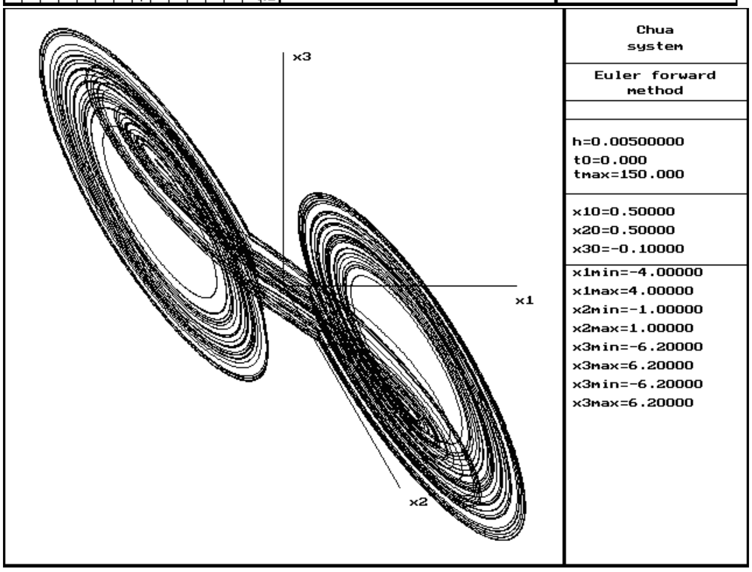
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x2
Chua
system
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method
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x1min=-4.00000
x1max=4.00000
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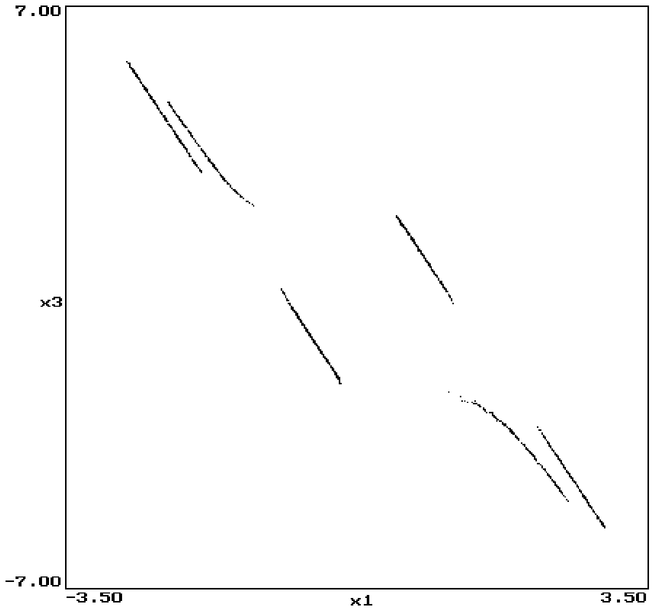


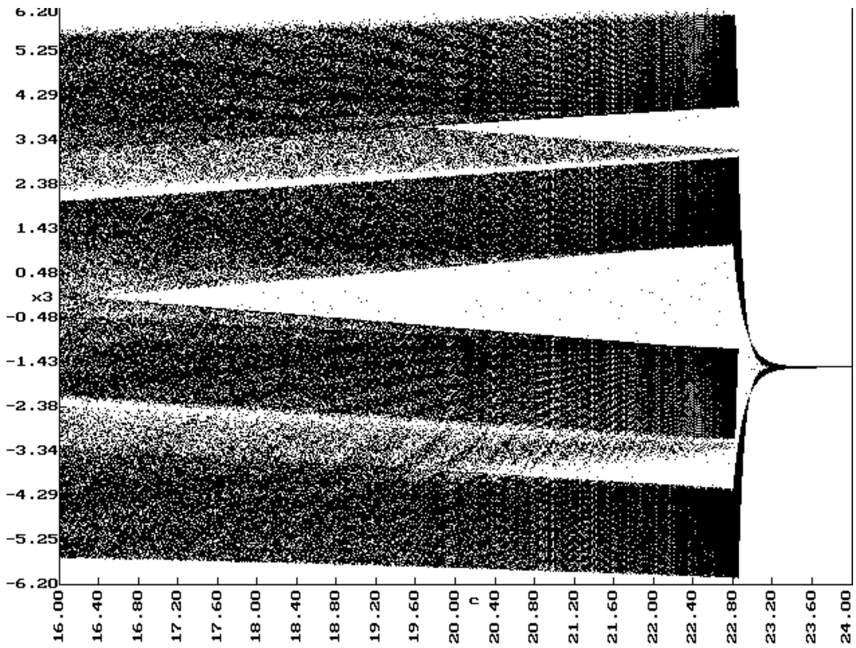




Fig.1 A chaotic trajectory of the generalized switch d.s. for $\beta = 15.7$: a) Phase portraits and time series. b) Three dimensional view. c) Poincaré section with the plane $x_2 = 0.1$. d) The bifurcation diagram.

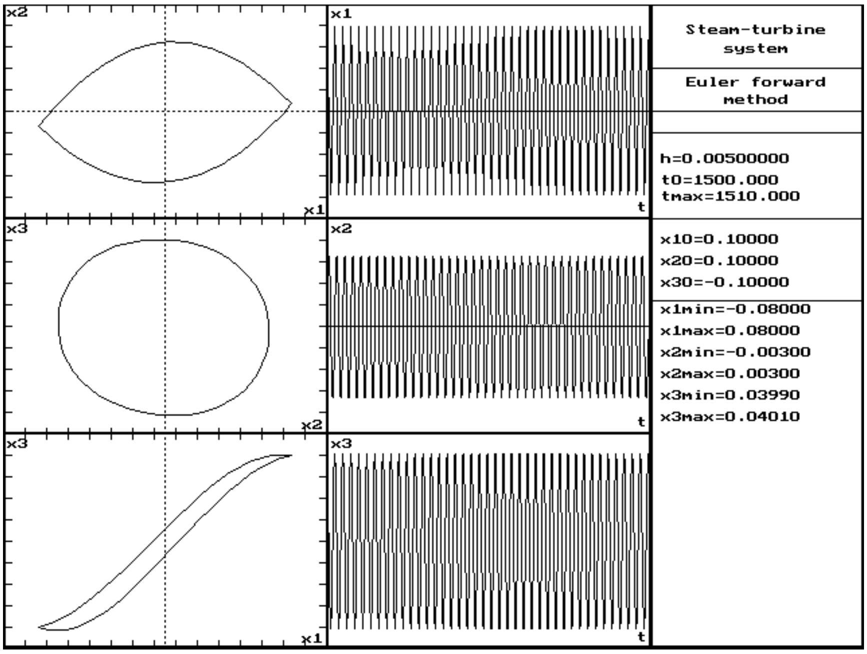
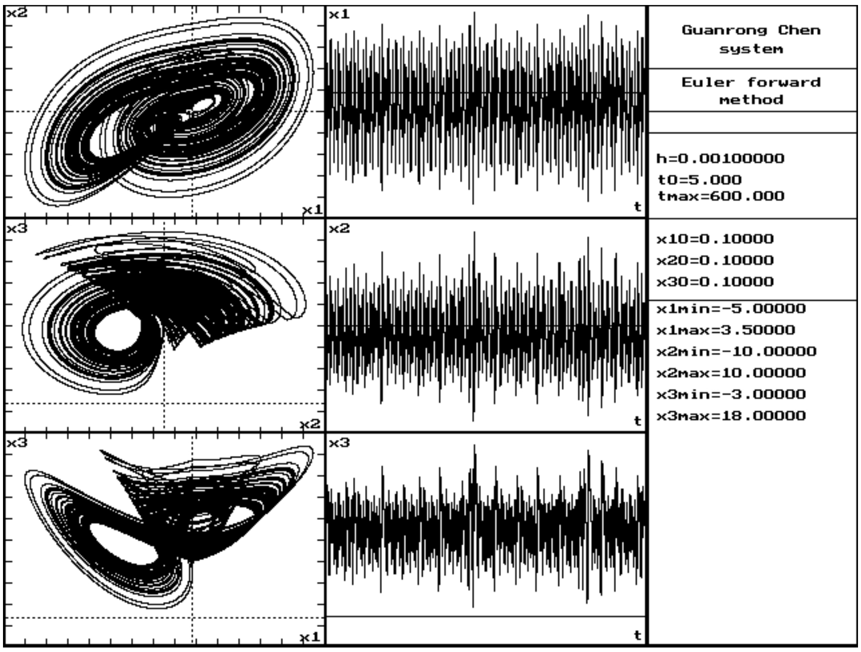
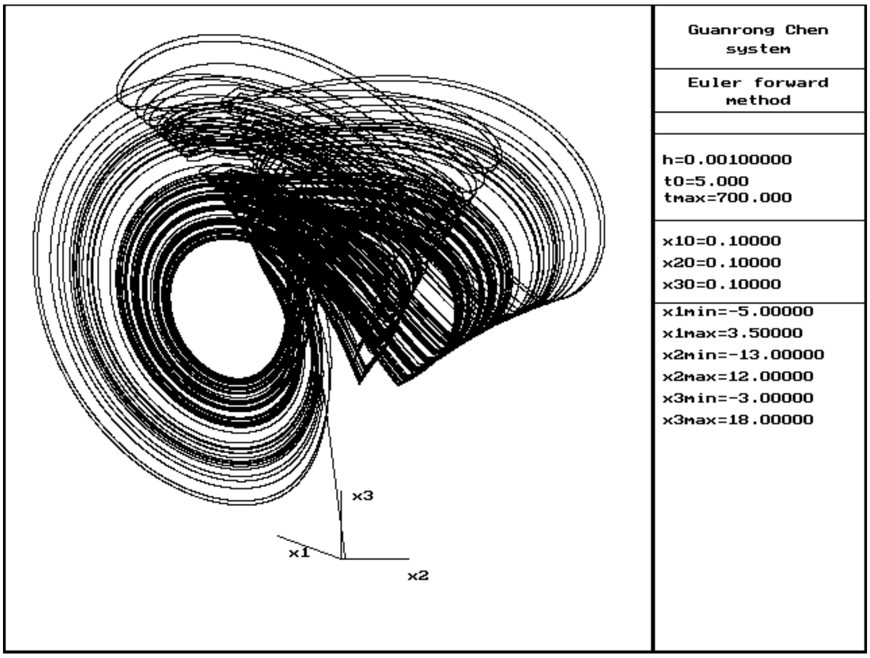



Fig.2 A periodic motion of the switch d.s. modeled by (9).
The first transient steps were omitted.




Guanrong Chen
system
Euler forward
method
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tmax=700.000
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x30=0.10000
x1min=-5.00000
x1max=3.50000
x2min=-13.00000
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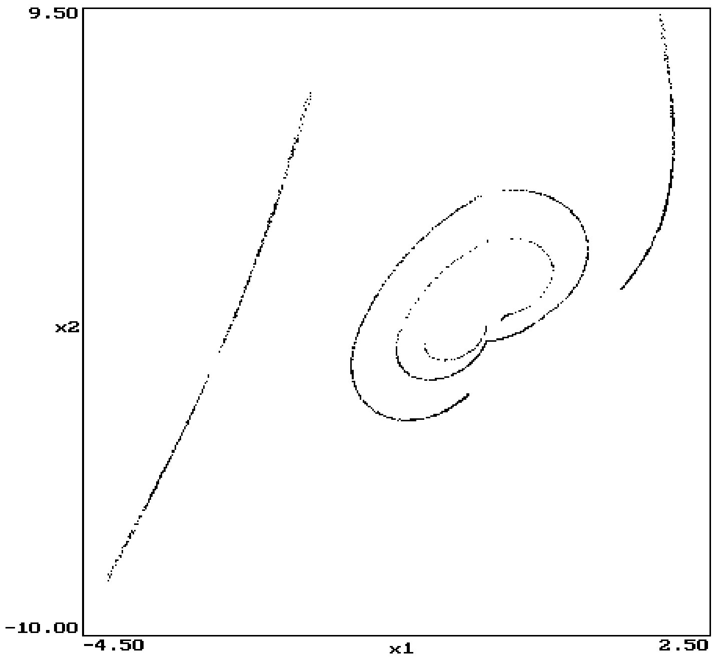
x3min=-3.00000
x3max=18.00000
x3
x1
x2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x2
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-4.50
x1
2.50

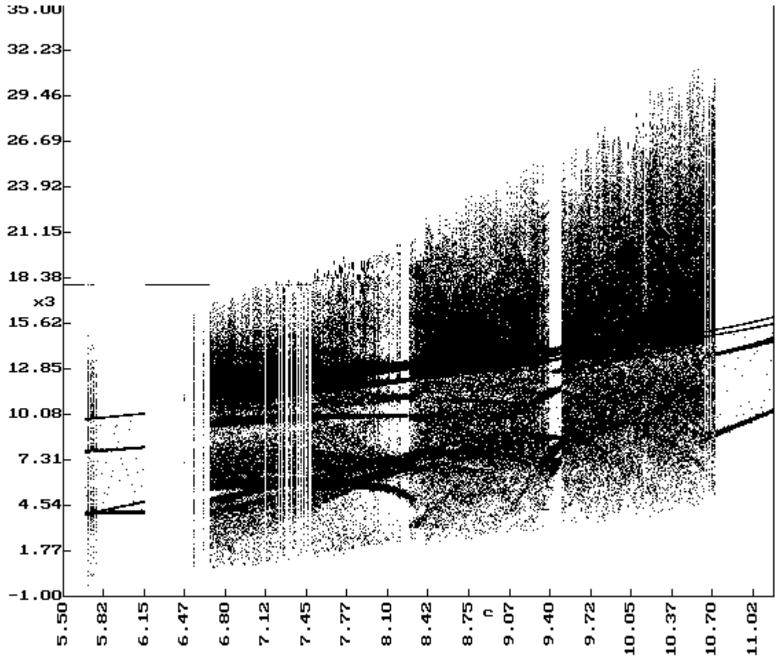



Fig.3. A chaotic trajectory of the discontinuous d.s. (10) for $c = 7$ : a) Phase portraits and time series. b) Three dimensional view. Poincaré section with the plane $x_3 = 10$ . d) Bifurcation scenario.

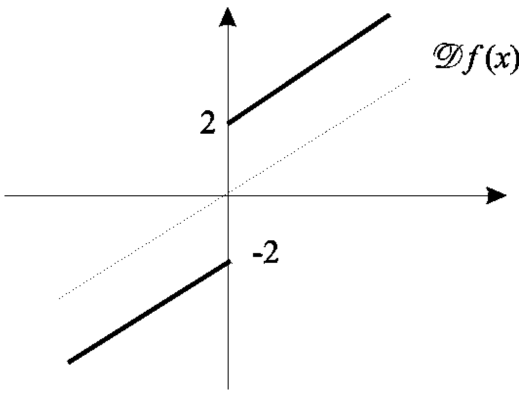
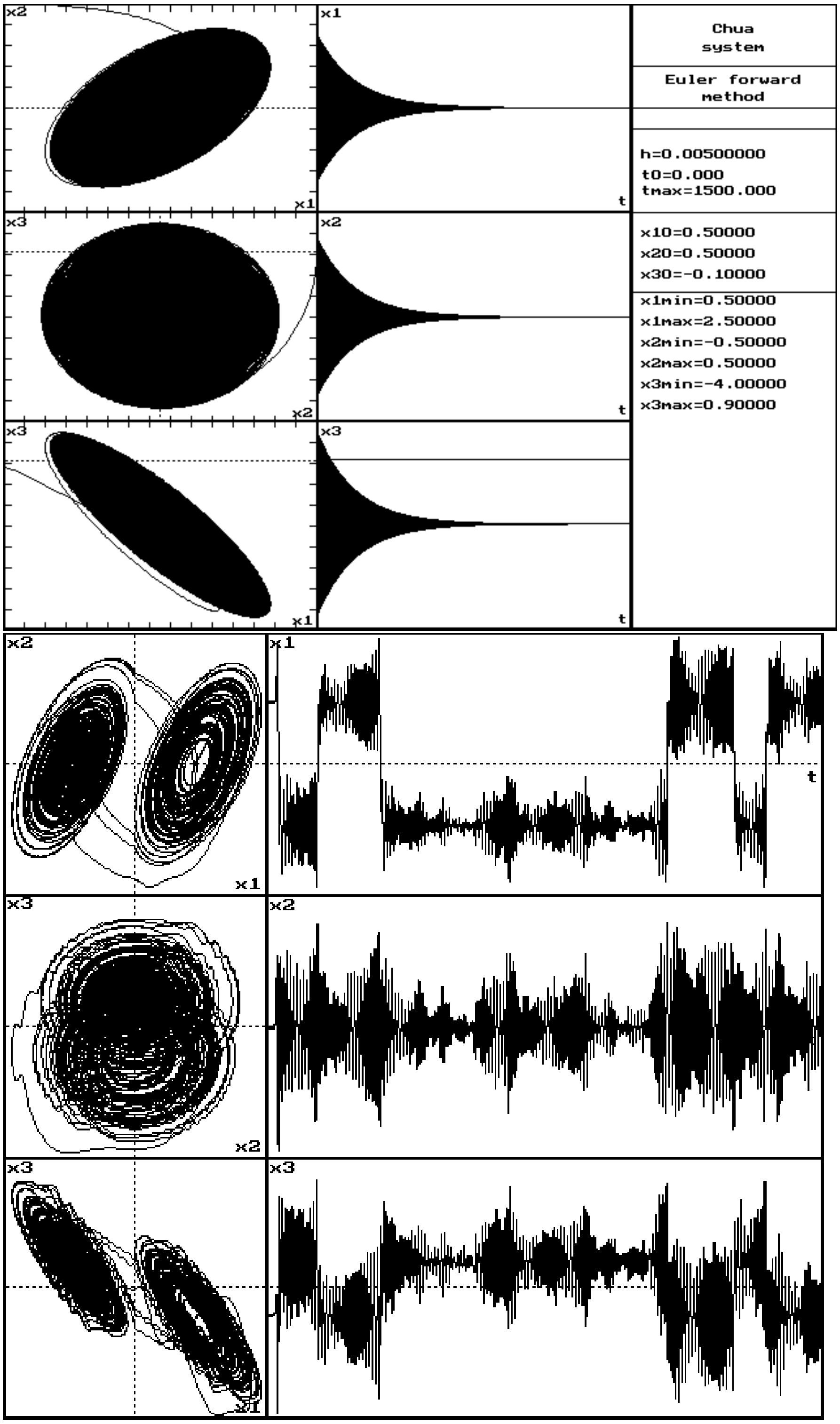



Fig.4. The graph of the generalized derivative of the function $f(x) = x - 2\,sgn(x)$.



Chua
system
Euler forward
method
h=0.00500000
t0=0.000
tmax=1500.000
x10=0.50000
x20=0.50000
x30=-0.10000
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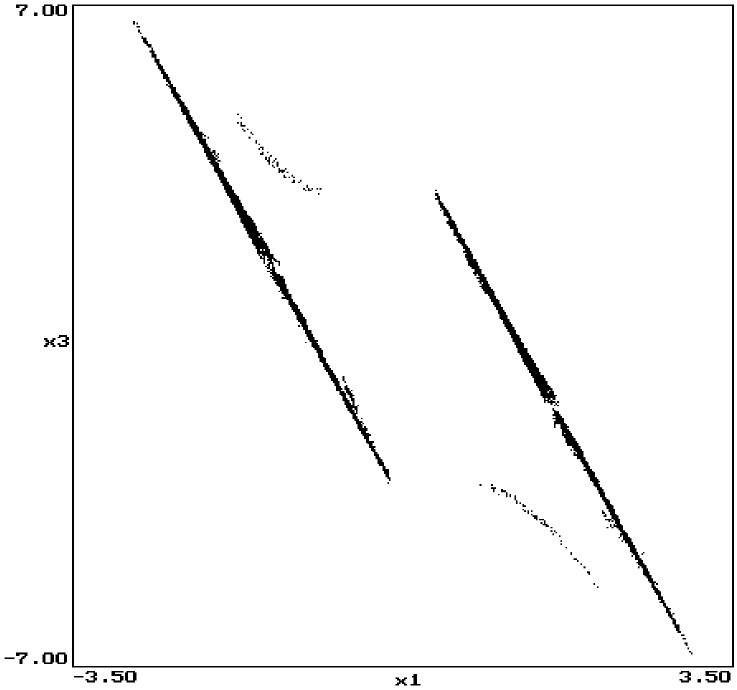
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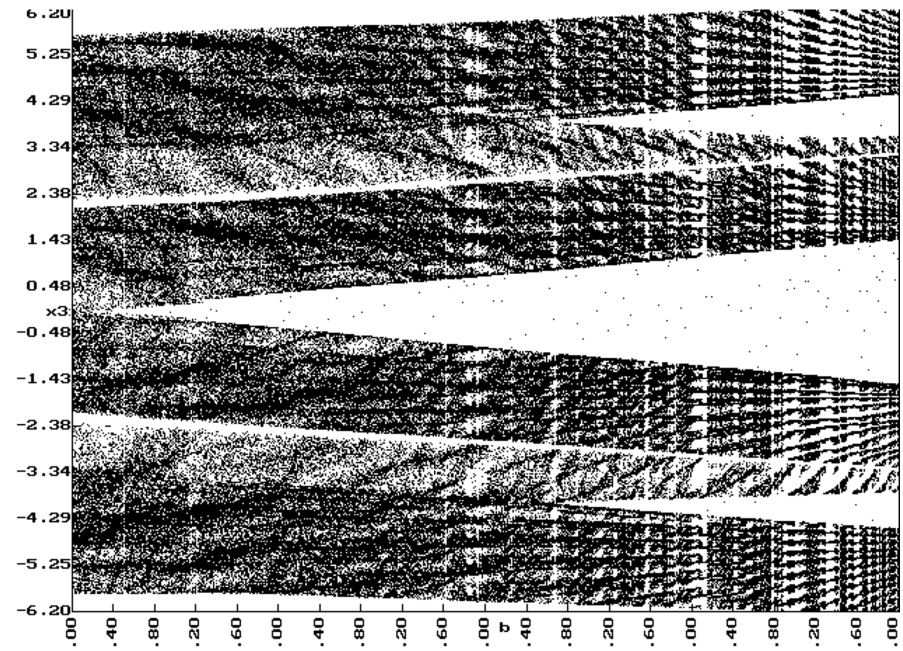
-3.50
x1
3.50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5.25
4.29
3.34
2.38
1.43
0.48
x3
-0.48
-1.43
-2.38
-3.34
-4.29
-5.25
-6.20
16.00
16.40
16.80
17.20
17.60
18.00
18.40
18.80
19.20
19.60
20.00
b
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22.00
22.40
22.80
23.20
23.60
24.00

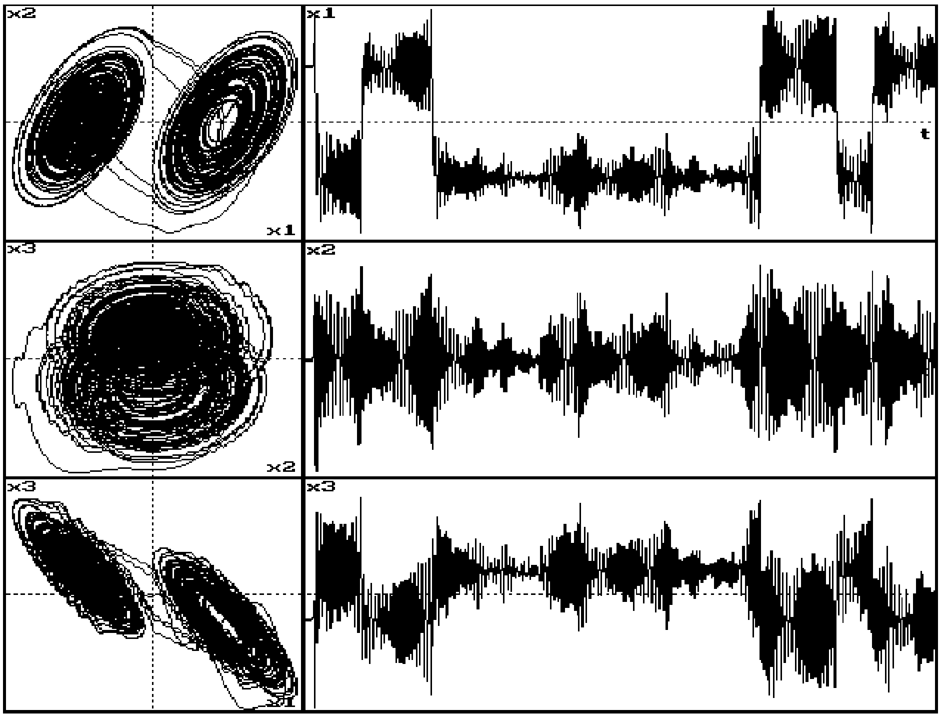
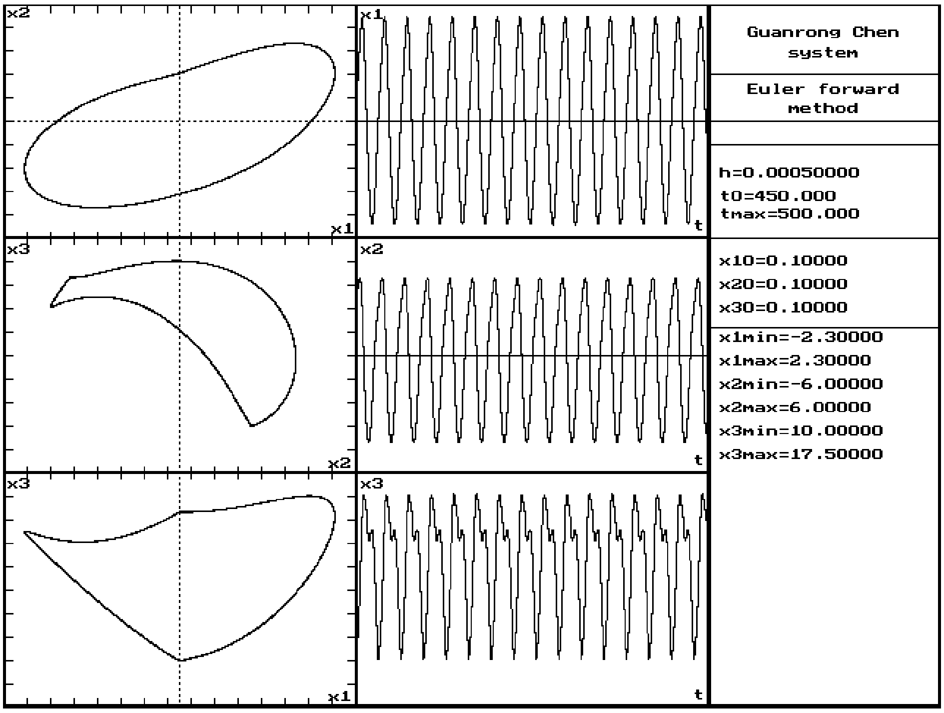


Fig.5. a) One of the stable fixed points of the uncontrolled generalized switch d.s. (7) for $\beta = 23.5$. b) A chaotic attractor of the controlled system, with $\varepsilon = 0.6$, $\sigma = 40$ and time delay $\tau = 1$. c) The Poincaré section. d) Bifurcation diagram. e) The double chaotic scroll attractor obtained using the anticontrol algorithm for $\varepsilon = 1.5$, $\sigma = 50$ and time delay $\tau = 1$.



x2
x1
x1
t
x3
x2
x2
t
x3
x1
x3
t
Guanrong Chen system
Euler forward method
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t0=450.000
tmax=500.000
x10=0.10000
x20=0.10000
x30=0.10000
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x1max=2.30000
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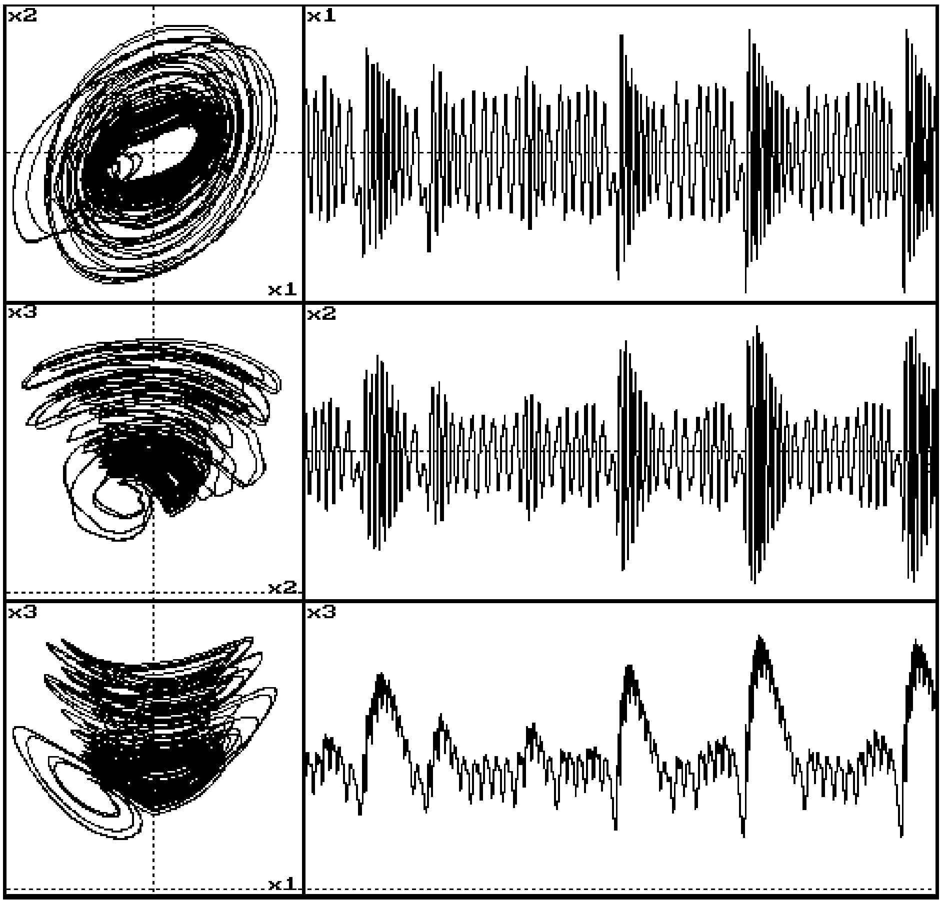
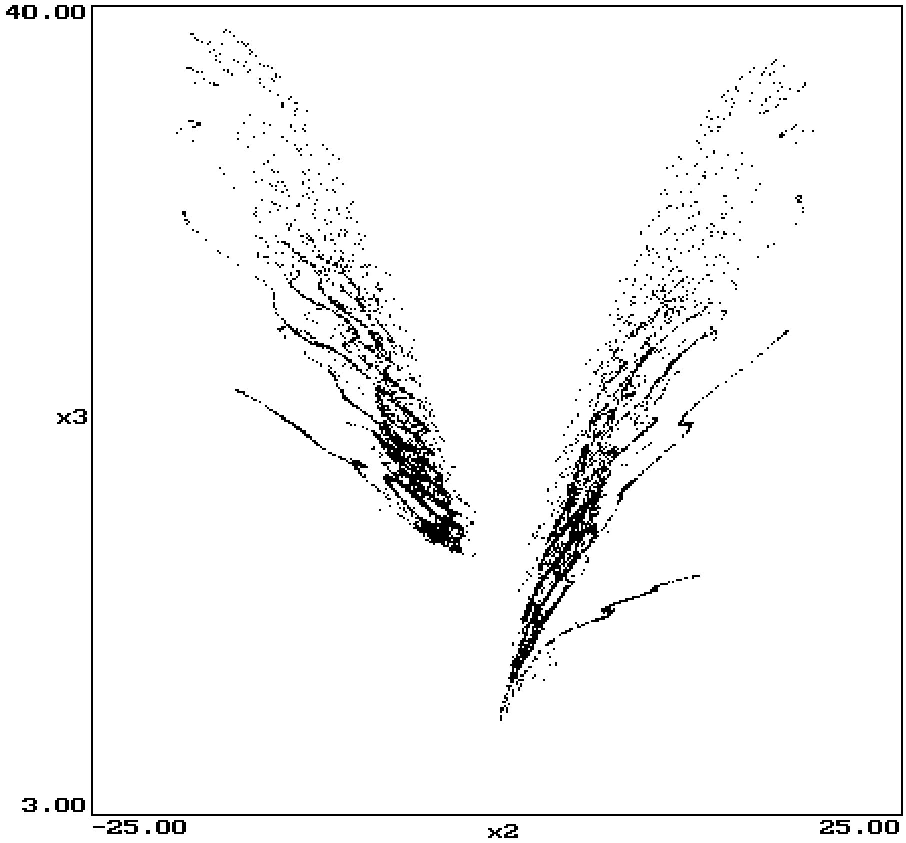

x2
x1
x1
x3
x2
x2
x3
x1
x3



40.00
x3
3.00
-25.00
x2
25.00

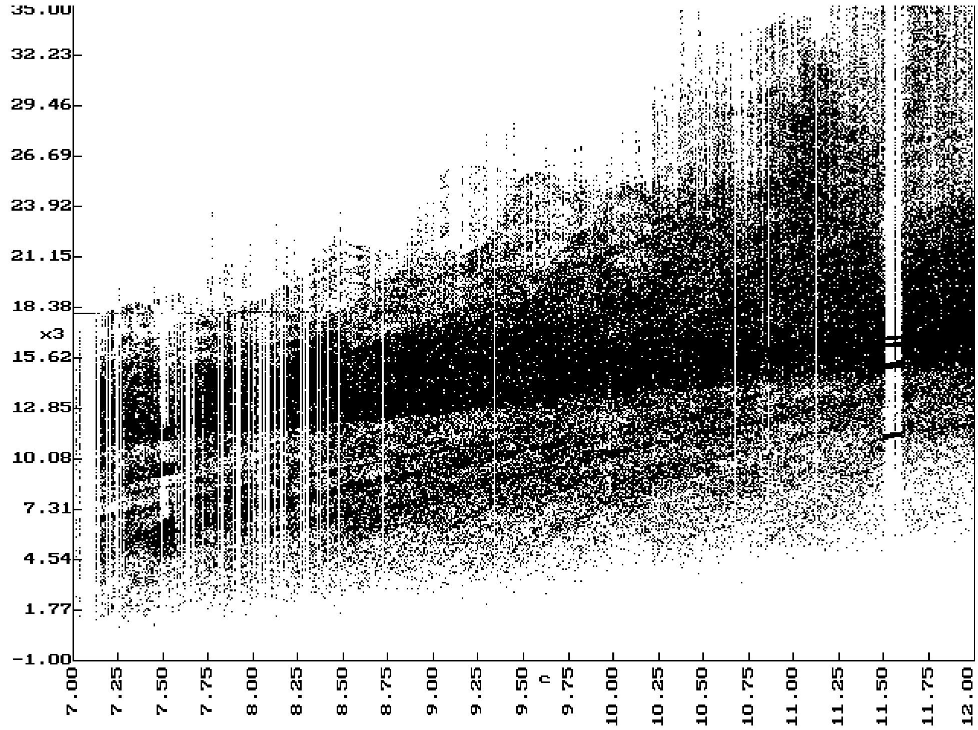



Fig. 6. a) periodic motion of the system (10) for $c = 11$. b) A chaotic attractor of the controlled system with $\delta\, c(t) = 2\, \sin\left(4\left(0.5\, x_1^2(t-1) - 1.18\, x_3(t-1)\right)\right)$. c) Poincaré section. d) Bifurcation scenario.

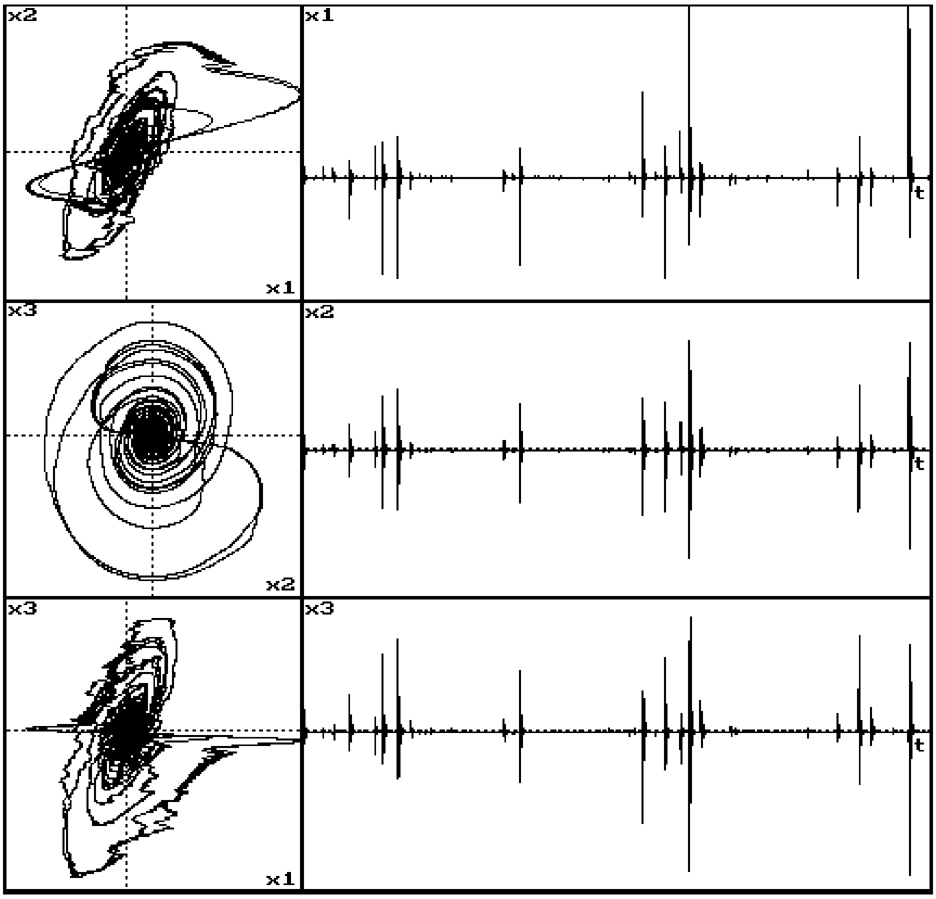


Fig. 7. A chaotic motion of the controlled system (9) obtained with $\delta\, a(t) = 15\, \sin\left(35\left(\, x_2(t-1)\right)\right).$