Attractor as a convex combination of a set of attractors

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Abstract

This paper presents an effective approach to constructing numerical attractors of a general class of continuous homogenous dynamical systems: decomposing an attractor as a convex combination of a set of other existing attractors. For this purpose, the convergent Parameter Switching (PS) numerical method is used to integrate the underlying dynamical system. The method is built on a convergent fixed step-size numerical method for ODEs. The paper shows that the PS algorithm, incorporating two binary operations, can be used to approximate any numerical attractor via a convex combination of some existing attractors. Several examples are presented to show the effectiveness of the proposed method.

Keywords: Parameter switching; Continuous-time system; Numerical attractor

1. Introduction

As is well known, due to the intrinsic difficulty or intractability, many studies on nonlinear dynamics rely on numerical analysis. Examples in point include invariant manifolds, basins of attractions, homoclinic and heteroclinic orbits, Smale horseshoes, and generally chaotic attractors. The present paper follows this traditional approach to carry out careful and subtle numerical analysis of complex attractors with respect to their structural composition and decomposition.

For general nonlinear dynamical systems, it is rarely possible to determine their attractors analytically. Therefore, numerical approximations constitute an important and natural part of a systematic analysis. If the ODEs describing a system have an attractor A, then the discrete dynamical system generated by some convergent numerical method can also have an attractor that converges to A [1] (see also [19], which suggests numerical methods that preserve qualitative properties of ODEs).

A major problem in the interpretation of the numerical results is that, in general, one cannot conclude the existence of a real attractor close to a numerical attractor [1].

On the other hand, under a variety of Lipschitz conditions, the Runge-Kutta methods define discrete dynamical systems. In [7, 8], the asymptotic behavior of the underlying dynamical system is compared with the asymptotic behavior of its numerical discretization.

The convergence can be studied either for finite-time evolution, when one concerns the global error, i.e. the study of the distance between the computed trajectory and the real trajectory, or can be proved for infinite-time evolution, when concerning whether the trajectories of numerical solutions converge to ω-limit sets of the dynamical system.

Many single-parameter chaotic dynamical systems, such as the Lorenz system, Rössler system, Chen system, Lotka–Volterra system, Rabinovich–Fabrikant (RF) system, Hindmarsh-Rose system, Lü system, etc. can be modeled as the following Initial Value Problem (IVP):

$$\dot{x}(t) = f(x(t)) := g(x(t)) + pBx(t), \quad x(0) = x_0, \tag{1}$$

where $t \in I = [0, T]$, $x_0 \in \mathbb{R}^n$, $p \in \mathbb{R}$ is the control parameter, $B \in \mathbb{R}^{n\times n}$ a constant matrix, and $g : \mathbb{R}^n \to \mathbb{R}^n$ a continuous nonlinear function. Because of the autonomous nature of system (1), hereafter the time variable t may not be explicitly indicated.

An example of dynamical systems modeled by the IVP (1) is the Lorenz system

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1), \\ \dot{x}_2 &= x_1(\rho - x_3) - x_2, \\ \dot{x}_3 &= x_1x_2 - \beta x_3, \end{aligned}$$

where $n = 3$ with $a = 10$ and $c = 8/3$, if one considers $p = \rho$ then system (1) has

$$g(x) = \begin{pmatrix} \sigma(x_2 - x_1) \\ -x_1x_3 - x_2 \\ x_1x_2 - \beta x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this paper one studies, for the first time, the possibility to approximate any numerical attractor of a given dynamical system depending on a single real parameter via a convex combination of some existing attractors. For this purpose, one considers a discrete fixed step-size numerical algorithm, the Parameter Switching (PS) algorithm, which switches the parameter p within a given finite set of parameter values. The algorithm is made on a finite single step-size convergent scheme for ODEs (here, the Standard Runge-Kutta method). Because the considered systems, modeled by the IVP (1) are autonomous, the resulting discrete time dynamical system modeled by the PS algorithm is also autonomous.

The convergence of the PS algorithm is presented in [4] (see also [5, 10]).

The algorithm can be used both for theoretical studies of dynamical systems modeled by a general class of systems, such as synchronization [13], chaos control and anticontrol, and as generalization of the Parrondo game (see e.g. [12, 14]). An experimental implementation shows that the PS algorithm works on real systems as well, e.g. electronic circuits [10].

Also, the PS algorithm can force the system to evolve along a desired attractor which, for some reasons, cannot be enhanced by setting the parameter with the corresponding

value p^0. Therefore, by switching the parameter with the PS algorithm while the IVP is integrated within a set of values, the generated attractor approximates the attractor corresponding to p^0 with sufficiently small error. However, the most important property induced by the PS algorithm is the possibility to express attractors as a convex combination of other attractors, which will be further explored in this paper.

The paper is organized as follows: Section 2 presents the PS algorithm and his numerical implementation, and in Section 3 attractors decomposition is expressed as a convex combination of a set of other attractors, while Section 4 presents several examples of the use of the PS algorithm to attractors decomposition. The Appendix presents a simple matlab code to implement the PS algorithm. The paper ends with the Conclusion section.

2. The PS algorithm

Throughout this paper, the following assumption is made.

H1 Function f in (1) is Lipschitz continuous.

Under **H1**, with an admissible initial condition x_0 for any p, the IVP (1) admits a unique and bounded solution.

Let a dynamical system be modeled by the IVP (1), with the initial condition x_0 fixed. Denote the set of admissible parameter values by $\mathcal{P}$, and a subset of $\mathcal{P}$ with $N > 1$ parameters $p_i \in \mathbb{R}$, $i = 1, 2, ..., N$, by $\mathcal{P}_N = \{p_1, p_2, ..., p_N\}$ (Fig. 1 (a)).

In this paper, by *attractor* one understands the *numerical attractor* obtained with some fixed time-step convergent numerical method (see e.g. [6, 7]), after transients are neglected [20].

Because of the solution uniqueness ensured by the Lipschitz continuity, to each $p_i \in \mathcal{P}_N$ there corresponds a unique attractor $A_i \in \mathcal{A}_N = \{A_1, A_2, ..., A_N\}$.

Hereafter, the set $\mathcal{P}_N$ is considered ordered: $p_1 < p_2 < ... < p_N$.

Given a suitable set of parameters $\mathcal{P}_N$ for a given $N > 1$, by switching periodically the value of the parameter p within the set $\mathcal{P}_N$ while the IVP (1) is numerically integrated with a fixed step-size numerical method for ODEs, the PS algorithm allows to approximate any attractor of system (1).

Suppose that one intends to generate some attractor A^o, which corresponds to $p := p^0$, but, it cannot be generated by integrating the IVP with $p = p^0$. There are many reasons for that in real dynamical systems. Thus, one may choose a set $\mathcal{P}_N = \{p_1, p_2, ..., p_N\}$, with $p_1 < p^0 < p_N$, such that the attractor A^0 can be numerically approximated by the attractor A^* generated using the PS method to be introduced bellow.

H2 A fixed step-size h explicit convergent numerical method (here the Runge-Kutta (RK) method) is used for solving the IVP (1) on the discrete nodes nh, $n = 1, 2, ...$

For a given step-size $h > 0$, the PS algorithm can be symbolized with the following scheme:

$$S := [m_1 \circ p_1, m_2 \circ p_2, ..., m_N \circ p_N]_h, \tag{2}$$

where $\mathcal{M}_N = \{m_1, m_2, ..., m_N\}$, $m_i \in \mathbb{N}^*$, $i = 1, 2, ..., N$, denotes the "weights" of the p values. More precisely, the term $m_i \circ p_i$ indicates the number of m_i times to specify the value of parameter p to be p_i.

The scheme (2) reads as follows: while the IVP (1) is numerically integrated with the fixed step-size method, for the first m_1 integration steps (i.e. $n = 1, 2, ..., m_1$), $p = p_1$; for the next m_2 steps (i.e. $n = m_1 + 1, m_1 + 2, ..., m_2$), $p = p_2$; and so on, till the last m_N steps, where $p = p_N$. Next, the algorithm repeats until the entire time integration interval is covered. The switching period of p, which is piece-wise constant, is $\sum_{i=1}^{N} m_i h$.

For simplicity of notation, hereafter the index h in (2) will be dropped.

The approximation of the attractor A^0 with the PS method, using some chosen sets $\mathcal{P}_N$ and $\mathcal{M}_N$, is obtained based on the following main result of the paper.

Theorem 1. *Given the IVP* (1)*, the numerical solution y_n, obtained with the PS method, approximates the numerical solution x_n, for $n = 1, 2, ...$, obtained with $p := p^0$, where*

$$p^0 = \frac{\sum_{i=1}^{N} m_i p_i}{\sum_{i=1}^{N} m_i}, \tag{3}$$

with the global error

$$e_n \leq nh\|B\|\|x_0\| \sum_{i=1}^{N} m_i |p_i - p^0| + \mathcal{O}(h).$$

Proof. See the proof of the numerical convergence in [4], the proof based on averaging theory in [2], or the analytical proof in [5]. □

As can be deduced from the proof presented in [2, 4, 5], the initial conditions for solutions x_n and y_n could be different within the same attraction basin.

In the following, the numerical solution obtained with the PS method and the reaching attractor, denoted A^*, are called *switched solution* and *switched attractor,* respectively, while the solution corresponding to $p := p^0$ and the reaching attractor, denoted A^0, are called *averaged solution* and *averaged attractor*, respectively.

Remark 1. *For a given N, the scheme* (2) *is usually not unique: there are several sets $\mathcal{M}_N$ and $\mathcal{P}_N$ which generate the same value of p^0 via formula* (3)*, but with different approximation precision.*

The following result can be directly derived from Theorem 1.

Corollary 2. *i) For every given sets $\mathcal{P}_N$ and $\mathcal{M}_N$, the switched attractor A^* approximates the averaged attractor A^0, denoted $A^* \approx A^0$, with p^0 given by* (3)*;*

ii) For each attractor A of the considered system (1)*, there exist sets $\mathcal{P}_N$ and $\mathcal{M}_N$, $N > 1$, such that A can be approximated by the PS method.*

As an example, suppose that one wants to approximate the stable cycle of the Lorenz system with the PS method corresponding to $p = 93$. In this case, the averaged attractor A^0 corresponds to $p^0 = 93$. Let, for example, $N = 3$. Then, in order to obtain in (3) $p^0 = 93$, a possible choice is $\mathcal{P}_3 = \{86, 95, 97\}$ and $\mathcal{M}_3 = \{2, 1, 3\}$, and the relation (3) reads $p^0 = (m_1p_1 + m_2p_2 + m_3p_3)/(m_1 + m_2 + m_3) = 93$ with the underlying scheme (2), $S = [2 \circ 86, 1 \circ 95, 3 \circ 97]$, which generates the switched attractor A^* as an approximation of the attractors A^0 for $p^0 = 93$.

Remark 2. *i) To check supplementarily the numerically match between the switched and averaged solutions, Hausdorff distance [15] [p. 114] has been utilized with the result being generally of order $1e-4$, or even smaller, depending on the h value.*

ii) The numerical implementation of the PS algorithm requires fixed step-size convergent numerical methods for ODEs (Standard Runge-Kutta method in this paper). Therefore, the approximation precision in Theorem 1 depends on the step-size h (see e.g. [11]). Also, special attention is needed regarding the initial conditions [5, 10].

To verify numerically the match between attractors A^* and A^0, a simple matlab code, where the PS algorithm uses the Standard Runge-Kutta method, is presented in Appendix for the particular case of the Lorenz system, which can be easily replaced with any other system in form (1). The code requires T_{max}, h, parameters $\mathcal{P}_N$, and weights $\mathcal{M}_N$. Using this code, one can easily test the algorithm performances, by adjusting h, N etc or the role of initial conditions. The output are the switched vector solution and the averaged solution.

$$[y, x] = PS(T_{max}, h, [m_1, m_2, ..., m_N], [p_1, p_2, ..., p_N], x_0, y_0).$$

Let implement numerically the scheme $S = [2 \circ 86, 1 \circ 95, 3 \circ 97]$, used before to approximate the attractor A^0 corresponding to $p^0 = 93$, with $h = 0.01$, and initial conditions $x_0 = [1, 1, 1]$ and $y_0 = (1.001, 1.001, 1.001)$, over the time interval $T_{max} = 1000$. The command line is

$$[y, x] = PS(1000, 0.01, [2, 1, 3], [86, 95, 97], [1, 1, 1], [1.001, 1.001, 1.001]).$$

The results are plotted in Fig. 2. In order to reveal the match between the switched attractor A^* (red plot) and the average attractor A^0 corresponding to $p^0 = 93$ (blue plot), the attractors are overplotted in the phase space (Fig. 2 (a)). The match between the two attractors can also be underlined by overplotting Poincaré sections (here, the cross section is $x_3 = 100$, Fig. 2 (b)), hystograms (Fig. 2 (c), time series (Fig. 2 (d)).

Remark 3. *The choice of $\mathcal{P}_N$ and $\mathcal{M}_N$ to obtain a set value p^0 is not unique. Therefore, for e.g. $N = 2$, the same attractor A^0, with $p^0 = 93$, can be obtained also with the simplest possible switching (alternating) scheme $S_1 = [1 \circ p_1, 1 \circ p_2]$, with, e.g., $p_1 = 92$ and $p_2 = 94$. Now, the algorithm acts by alternating p, at each step, between p_1 and p_2. Also, this attractor can be obtained with the scheme $S_2 = [3 \circ p_1, 3 \circ p_2]$, for $p_1 = 92$ and $p_2 = 94$, or $[2 \circ p_1, 2 \circ p_2]$, for $p_1 = 91$ and $p_2 = 95$.*

Chaotic attractors can also be approximated with the PS algorithm. For example, suppose that one wants to approximate the chaotic attractor corresponding to $p^0 = 25.5$. Using e.g. the scheme $S = [2 \circ p_1, 1 \circ p_2]$ with $p_1 = 24$ and $p_2 = 28.5$, one obtains $p^0 = (2p_1 + p_2)/3 = 25.5$. The result, for initial conditions $(1, 1, 1)$ and $(1.001, 1.001, 1.001)$, is presented in Fig. 3. It should be noted that, contrarily to the regular cases where after some transients A^* approaches quickly A^0, in the case of chaotic attractors, dependending on initial conditions the approximation is gradual.

Also, the case of approximating attractive equilibria can be easily implemented. For example, to approximate the stable equilibrium corresponding to $p^0 = 10$, one can use the scheme $S = [1 \circ p_1, 1 \circ p_2]$ with $p_1 = 9$ and $p_2 = 11$.

The inherent numerical limitations of the PS algorithm (such as relative large m values, large distances between values of p within $\mathcal{P}_N$ or dependence on h) are presented in [11].

3. Attractor properties induced by the PS algorithm within the set $\mathcal{A}_N$

The main characteristic of the PS algorithm relies on the convex relation (3)

Proposition 3. *p^0 is a convex combination of the elements of the set $\mathcal{P}_N$.*

Proof. By denoting $\alpha_j = m_j / \sum_{i=1}^{N} m_i$, $j = 1, 2, ..., N$, relation (3) becomes

$$p^0 = \sum_{i=1}^{N} \alpha_i p_i, \tag{4}$$

with $\sum_{i=1}^{N} \alpha_i = 1$. □

Hereafter, unless specified, for simplicity the averaged attractor A^0 corresponding to some value $p^0 = p$ which will be approximated, will be denoted A_p.

As seen in Section 2, the stable cycle of the Lorenz system, A_{93}, can be approximated by the numerical attractor generated, e.g. for $N = 3$, with $\mathcal{P}_3 = \{86, 95, 97\}$ and $\mathcal{M}_3 = \{2, 1, 3\}$, for which $\alpha_1 = 2/6 = 1/3$, $\alpha_2 = 1/6$ and $\alpha_3 = 3/6 = 1/2$. In this case, the relation (3) gives, the value $p^0 = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 = 93$ with the scheme $S = [2 \circ 86, 1 \circ 95, 3 \circ 97]$.

Also, for $N = 2$, the simplest possible switching scheme is $[1 \circ p_1, 1p_2]$, with $\mathcal{P}_2 = \{92, 94\}$, $\alpha_1 = \alpha_2 = 1/2$. Also, A_{93} can be approximated by the scheme $S = [3 \circ p_1, 3 \circ p_2]$, with $p_1 = 92$ and $p_2 = 94$, when $\alpha_1 = 3/6 = 1/2$, and $\alpha_2 = 3/6 = 1/2$, or by the scheme $S = [2 \circ p_1, 2 \circ p_2]$, with $p_1 = 91$ and $p_2 = 95$, when $\alpha_1 = \alpha_2 = 2/4 = 1/2$.

Proposition 4. *The sets $\mathcal{P}_N$ and $\mathcal{A}_N$ are order isomorphic and the synthesized attractor A^* verifies the relation*

$$A_1 \prec A^* \prec A_N. \tag{5}$$

Proof. Under Assumption **H2**, it is natural to assume that there exists a linear (bijective) order-preserved mapping

$$H : \mathcal{P} \to \mathcal{A}, A = H(p). \tag{6}$$

Note that the bijective map H maps any intervals of the parameter axis $\mathcal{P}$ to intervals on the attractors axis $\mathcal{A}$. Consider the restriction $H : \mathcal{P}_N \to \mathcal{A}_N$ for $\mathcal{P}_N \subset \mathcal{P}$ and $\mathcal{A}_N \subset \mathcal{A}$. To the interval $[p_1, p_N]$, with $p_1, p_N \in \mathcal{P}_N$, corresponds the interval $[A_1, A_N]$, with $A_1, A_N \in \mathcal{P}_N$. If one considers the set $\mathcal{P}$ endowed with the total order $<$, the induced total order in $\mathcal{A}$, denoted by $\prec$, implies that $p_i < p_j$ if and only if $A_i \prec A_j$, for all $i, j \in \{1, 2, ..., N\}$. Then, there exists an order isomorphism from $(\mathcal{P}_N, <)$ to $(\mathcal{A}_N, \prec)$, such that for every p_i and p_j, $i, j \in \{1, 2, ..., N\}$, $p_i < p_j$ if and only if $H(p_i) \prec H(p_j)$. Because from (3), $p_1 < p^0 < p_N$, it follows that $A_1 \prec A^0 \prec A_N$ and by Corollary 2 i), $A_1 \prec A^* \prec A_N$ (see Fig. 1 (b) where, for $N = 5$, the set $\mathcal{P}_N$ is represented schematically on the axis of the set of parameters p, $\mathcal{P}$ and $\mathcal{A}_N$ on the axis of attractors $\mathcal{A}$). □

Note that, by the convex relation (3), and from the bijectivity of H, A^0 is different from A_i, $i = 1, 2, ..., N$.

Next, on the set $\mathcal{A}$, introduce two binary relations (operators) $(\mathcal{A}, \oplus, \otimes)$, with $\oplus$ being *addition of attractors* and $\otimes$ being *multiplication of attractors by positive real numbers.*

By Proposition 3 and because H is defined as a general mapping from parameters p_i to the corresponding attractors, the following result presents a way to describe the averaged attractor.

Corollary 5. *For given sets $\mathcal{P}_N$ and $\mathcal{M}_N$, the average attractor A^0, corresponding to p^0 given by* (3)*, can be expressed as*

$$A^0 = \alpha_1 \otimes A_1 \oplus \ldots \oplus \alpha_N \otimes A_N. \tag{7}$$

Proof. A general way of defining $\oplus$ and $\otimes$ is

$$\alpha \otimes A := H(\alpha H^{-1}(A)), \tag{*}$$

and

$$A_1 \oplus A_2 := H(H^{-1}(A_1) + H^{-1}(A_2)), \tag{**}$$

whenever $H^{-1}(A_1) + H^{-1}(A_2) \in \mathcal{P}$ and $0 < \alpha < 1$.

Consider $0 < \alpha_i < 1$, for $i = 1, 2, \ldots, N$, satisfying (4). Then, because $H^{-1}(A_i) = p_i$, $i \in \{1, 2, \ldots, N\}$, one has

$$\begin{aligned}
&\alpha_1 \otimes A_1 \oplus \alpha_2 A_2 \oplus \ldots \oplus \alpha_N \otimes A_N = \\
&H(\alpha_1 H^{-1}(A_1)) \oplus H(\alpha_2 H^{-1}(A_2)) \oplus \ldots \oplus H(\alpha_N H^{-1}(A_N)) = \\
&H(\alpha_1 p1) \oplus H(\alpha_2 p_2) \oplus \ldots \oplus H(\alpha_N p_N).
\end{aligned}$$

Next, for $p_i \in \mathcal{P}_N$, $\alpha_i p_i \in \mathcal{P}$, $H(\alpha_i p_i) \in \mathcal{A}$. Using the relation (**) and because $H^{-1}(H(\alpha_i p_i)) = \alpha_i p_i$, for $i = 1, 2, \ldots, N$, one obtains

$$\begin{aligned} &H(\alpha_1 p1) \oplus H(\alpha_2 p_2) \oplus \ldots \oplus H(\alpha_N p_N) = \\ &H(H^{-1}(H(\alpha_1 p_1)) + \ldots + H^{-1}(H(\alpha_N p_N))) = \\ &H(\alpha_1 p_1 + \ldots + \alpha_N p_N). \end{aligned}$$

Using the expression (4) for p^0, one has

$$\alpha_1 \otimes A_1 \oplus \alpha_2 A_2 \oplus ... \oplus \alpha_N \otimes A_N = H(p^0) = A^0.$$

□

Note that the decomposition (7) depends only on the inherent numerical errors and not on $\mathcal{P}_N$ and $\mathcal{M}_N$, or N (see Remark 3) in the following sense: the approximated attractors A^0 are similar for whatever choice of N and the sets $\mathcal{P}_N$ and $\mathcal{M}_N$, up to the inherent numerical errors.

Besides the commutativity and the associativity of $\oplus$, properties that can be easily proved, the following commutativity property is useful for applications.

Proposition 6. *For $A_1, A_2 \in \mathcal{A}$ and $\alpha_1, \alpha_2 \in (0, 1)$,*

$$\alpha_1 \otimes A_1 \oplus \alpha_2 \otimes A_2 = \alpha_2 \otimes A_2 \oplus \alpha_1 \otimes A_1. \tag{8}$$

For $N = 2$, the relation (8) under the PS algorithm means $[m_1 \circ p_1, m_2 \circ p_2] = [m_2 \circ p_2, m_1 \circ p_1]$, an equality that can be verified via the code in Appendix. Generalization for $N > 2$ also holds and can be verified numerically.

Remark 4. *Relation* (7) *shows that for every attractor A there exists a set $\mathcal{A}_N = \{A_1, A_2, ..., A_N\}$ which could be considered as an "ordered basis", where A can be decomposed. However, beside the spanning property, ensured by* (7)*, the set $\mathcal{A}_N$ must verify linear independency: for every set $\mathcal{A}_N$ and set $\mathcal{M}_N$, if $\alpha_1 \otimes A_1 \oplus \alpha_2 \otimes A_2 \oplus ... \oplus \alpha_N \otimes A_N = 0$, where 0 can be considered the empty set, then $\alpha_1 = \alpha_2 = ... = \alpha_N = 0$. However if all coefficients α are null, the PS algorithm cannot be applied.*

4. Examples

In this section, besides the Lorenz system, examples of the utilization of the PS algorithm to approximate and decompose attractors of three systems are considered.

- Let the averaged attractor $A^0 = A_{93}$ of the Lorenz system be obtained by the scheme $[m_1 \circ p_1, m_2 \circ p_2]$ with $m_1 = 1$, $m_2 = 1$, i.e. $\alpha_1 = \alpha_2 = 1/2$, $p_1 = 92$, and $p_2 = 94$. Then,

$$\tfrac{1}{2} \otimes A_{92} \oplus \tfrac{1}{2} \otimes A_{94} \overset{(*)}{=} H(\tfrac{1}{2}H^{-1}(A_{92})) \oplus H(\tfrac{1}{2}H^{-1}(A_{94})) = H\big(\tfrac{92}{2}\big) \oplus H\big(\tfrac{94}{2}\big) := A^{'} \oplus A^{''} \overset{(**)}{=}$$

$$H[H^{-1}(A^{'}) + H^{-1}(A^{''})] = \big[H^{-1}\big(H\big(\tfrac{92}{2}\big)\big) + H^{-1}\big(H\big(\tfrac{94}{2}\big)\big)\big] = H\big(\tfrac{92}{2} + \tfrac{94}{2}\big) = H(93) = A_{93},$$

where $A^{'}, A^{''} \in \mathcal{A}$, i.e.

$$A_{93} = \frac{1}{2} \otimes A_{92} \oplus \frac{1}{2} \otimes A_{94}.$$

- Because the scheme (3) is not unique, A_{93} can also be obtained with the scheme $[m_1 \circ p_1, m_2 \circ p_2, m_3 \circ p_3]$ and, therefore, can be decomposed as

$$A_{93} = H\big(\tfrac{1}{3}86 + \tfrac{1}{6}95 + \tfrac{1}{2}97\big) = \tfrac{1}{3} \otimes H(86) \oplus \tfrac{1}{2} \otimes H(95) \oplus \tfrac{1}{2}H(97) =$$

$$\tfrac{1}{3} \otimes A_{86} \oplus \tfrac{1}{6} \otimes A_{95} \oplus \tfrac{1}{2} \otimes A_{97}.$$

Table 1 presents three more examples of systems modeled by the system (1). For each system, two significant cases are considered: a stable cycle and a chaotic attractor. $T_{max} = 1000$ for cycles and $T_{max} = 1500$ for chaotic attractors. In order to underline the efficacy of the PS method, a system with strong nonlinearity (the RF system) is considered. Also, as is known, the Rössler system presents a Z-shaped slow manifold in its phases space where the motion is slow until an edge is reached whereupon the trajectory jumps to the other branch of the manifold [18], representing a good test for the PS algorithm. In addition to the RF system, where $h = 0.001$ (see e.g. [17] for the special requirements related to h, initial conditions, utilized numerical method, parameters used to integrate this system), for the other systems, $h = 0.01$. Also, p^0 represents the parameter value whose underlying attractor A^0 will be decomposed. The plots represent the superimposed images of the approximated attractor A^0 and the switched attractor A^*. Note that, generally, systems modeled by (1) allow different choices of p (here, for the Rössler system and the RF system).

- The system modeling a cancer tumor is described by the following equations [16]:

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_1) - x_1x_2 - 2.5x_1x_3, \\ \dot{x}_2 &= 0.6x_2(1 - x_2) - 1.5x_1x_2, \\ \dot{x}_3 &= 4.5\tfrac{x_1x_3}{x_1+1} - 0.2x_1x_3 - px_3, \end{aligned}$$

which belongs to the classes defined by the IVP (1) with

$$g(x) = \begin{pmatrix} x_1(1 - x_1) - x_1x_2 - 2.5x_1x_3 \\ 0.6x_2(1 - x_2) - 1.5x_1x_2 \\ \frac{x_1x_3}{x_1+1} - 0.2x_1x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

a) The stable cycle $A_{0.555}$ can be decomposed with the scheme $[2 \circ 0.198, 2 \circ 0.302, 1 \circ 0.402, 4 \circ 0.710, 3 \circ 0.806]$ as follows (Table 1, first row):

$$A_{0.555} = \frac{1}{6} \otimes A_{0.198} \oplus \frac{1}{6} \otimes A_{0.302} \oplus \frac{1}{12} \otimes A_{0.402} \oplus \frac{1}{3} \otimes A_{0.702} \oplus \frac{1}{2} A_{0.806}.$$

b) The chaotic attractor corresponding to A_{544} with the scheme $[1 \circ 0.535, 1 \circ 0.553]$ can be decomposed in the following form (Table 1, second row):

$$A_{0.544} = \frac{1}{2} \otimes A_{0.535} \oplus \frac{1}{2} \otimes A_{0.553}.$$

- Consider the Rössler system

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3, \\ \dot{x}_2 &= x_1 + a x_2, \\ \dot{x}_3 &= b + x_3(x_1 - p), \end{aligned}$$

with $a = b = 0.1$. Note that p could also be chosen to be a.

$$g(x) = \begin{pmatrix} -x_2 - x_3 \\ x_1 + 0.1 x_2 \\ 0.1 + x_1 x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

a) Consider the stable cycle $A_{8.5}$. With the scheme $[2 \circ 8.28, 1 \circ 8.41, 2 \circ 8.540, 3 \circ 8.65]$, the attractor can be decomposed in the following form (Table 1, third row):

$$A_{8.5} = \frac{1}{4} \otimes A_{8.28} \oplus \frac{1}{8} A_{8.41} \oplus \frac{1}{4} \otimes A_{8.54} \oplus \frac{3}{8} \otimes A_{8.65}.$$

b) A chaotic attractor corresponding to $p = 18$ can be generated with the scheme $[1 \circ 17, 1 \circ 19]$ and decomposed as (Table 1, 4th row):

$$A_{18} = \frac{1}{2} \otimes A_{17} \oplus \frac{1}{2} \otimes A_{19}.$$

- The RF system [17], with extremely rich dynamics,

$$\begin{aligned} \dot{x}_1 &= x_2(x_3 - 1 + x_1^2) + a x_1, \\ \dot{x}_2 &= x_1(3x_3 + 1 - x_1^2) + a x_2, \\ \dot{x}_3 &= -2x_3(p + x_1 x_2), \end{aligned}$$

where $a = 0.1$. Note that p could also be a.

Now,

$$g(x) = \begin{pmatrix} x_2(x_3 - 1 + x_1^2) + 0.1x_1 \\ x_1(3x_3 + 1 - x_1^2) + 0.1x_2 \\ -2x_1x_2x_3) \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

a) A stable cycle $A_{0.265}$ can be obtained with the scheme $[1 \circ 0.2615, 2 \circ 0.2642, 3 \circ 0.2667]$, with decomposition (Table 1, 5th row)

$$A_{0.265} = \frac{1}{6} \otimes A_{0.2615} \oplus \frac{1}{3} \otimes A_{0.2642} \oplus \frac{1}{2} A_{0.2667}.$$

b) The chaotic $A_{0.2715}$, which is hidden [17], with the scheme $[1 \circ 0.2714, 1 \circ 0.2716]$, has the following decomposition (Table 1, 6th row):

$$A_{0.2715} = \frac{1}{2} \otimes A_{0.2714} \oplus \frac{1}{2} A_{0.2716}.$$

Conclusion and discussion

In this paper, it is shown that the PS algorithm allows to numerically approximate attractors of systems modeled by the IVP (1). The algorithm is useful especially for practical problems, when some parameter values cannot be directly determined. With the PS algorithm, one can approximate any desired unknown attractor. Based on the PS algorithm, the attractors can be expressed as a convex combination of a set of some existing attractors.

Depending on the type of attractors $A_i \in \mathcal{A}_N$, $i = 1, 2, ..., N$, the attractor A^0 can be decomposed as a function of either chaotic attractors, regular attractors, or regular and chaotic attractors, and the decomposition (7) can be used to generalize Parrondo's paradox. For example, because for the values of the set $\mathcal{P}_3 = \{86, 95, 97\}$, the Lorenz system evolves chaotically. If one denotes with $chaos_i$, $i = 1, 2, 3$, the behaviors corresponding to $\mathcal{P}_3$, and with *order* the generated motion corresponding to $p^0 = 92$, generated with the PS algorithm, then by the scheme $S = [2 \circ 86, 1 \circ 95, 3 \circ 97]$, one obtains

$$order = \frac{1}{3} \otimes chaos_1 \oplus \frac{1}{6} \otimes chaos_2 \oplus \frac{1}{2} \otimes chaos_3.$$

For $N = 2$, the relation (7) can be viewed as a generalization of Parrondo's paradox, $winning = losing + losing$, if one replaces *winning* with *order* and *chaos* with *losing*, or also as a control-like algorithm induced by the PS method. The reverse form obtained with the PS algorithm, $chaos = order_1 \oplus order_2$, is an anticontrol-like algorithm (see [12, 14], where generalizations of Parrondo's game obtained with the PS algorithm are presented). The quantities α_i, $i = 1, 2, 3$, in (7) could be considered as weights like of the attractors in the decomposition of A^0.

Further studies will be focused on reducing the approximation errors, e.g. by using other more accurate fixed step-size numerical schemes for solving ODEs. Studying the applicability of the PS algorithm to other more general classes of systems will be another interesting task.

In this paper, the focus is on periodic PS, but one could also study quasi-periodic PS for instance with $p_i = c\alpha_i$ for a positive constant c, irrational α and $\{.\}$ is the fractional part function.

Another approach is to generate p_i by a function $p_{i+1} = g(p_i)$ for a continuous function $g : [0, 1] \to [0, 1]$, for instance a logistic map i.e. a skew-product discrete system.

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Appendix

Matlab code for the PS method

```
function [y,x]=PS(Tmax,h,M,P,x0,y0)
%Input:
%Tmax: integration interval;
%h: step-size;
%M: weights;
%P: parameterS set;
%x0, y0 initial conditions of A^0 and A^* (column vectors);
%Note that lenght(M)=lenght(P)=N;
%Output:
%y: A^0 vector;
%x: A^* vector;
%Example:
%[y,x]=PS
   (500,0.001,[2,1,3],[86,95,97],[1;1;1],[1.001;1.001;1.001]);

n=round(Tmax/h);
x=zeros(3,n);%3 is the system dimension in the phase space
y=zeros(3,n);

%Switched attractor A^*
i=1;j=1;
```

```
l=length(P);
y(:,1) = y0;
while j<n
    for k=1:M(i)
        K_1 = h*f(y(:,j),P(i));
        K_2 = h*f(y(:,j) + 1/2*K_1,P(i));
        K_3 = h*f(y(:,j) + 1/2*K_2,P(i));
        K_4 = h*f(y(:,j) + K_3,P(i));
        y(:,j+1) = y(:,j) + 1/6*(K_1 + 2*K_2 + 2*K_3 + K_4);
        j=j+1;
    end
    i=mod(i,l);
    i=i+1;
end;

%Average attractor A^0
p1=cumprod([M;P]);
p0=sum(p1(2,:))/sum(M);%p^0
j=1;
x(:,1) = x0;
while j<n
    K_1 = h*f(x(:,j),p0);
    K_2 = h*f(x(:,j) + 1/2*K_1,p0);
    K_3 = h*f(x(:,j) + 1/2*K_2,p0);
    K_4 = h*f(x(:,j) + K_3,p0);
    x(:,j+1) = x(:,j) +  1/6*(K_1 + 2*K_2 + 2*K_3 + K_4);
    j=j+1;
end

function der = f(u,p)% Lorenz system
der=zeros(3,1);
der(1) =10*(u(2)-u(1));
der(2) = u(1)*(p-u(3))-u(2);
der(3) = u(1)*u(2)-8/3*u(3);
```

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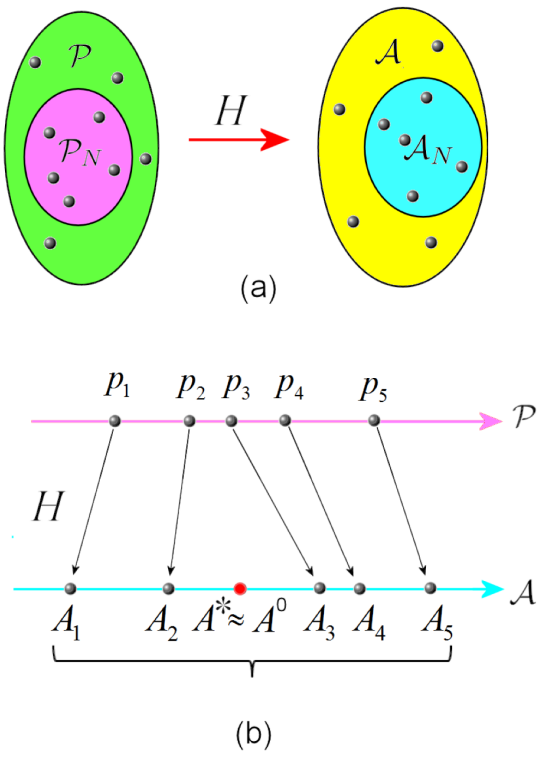


Figure 1: (a) Sets $\mathcal{P}$ and $\mathcal{P}_N$, and the corresponding sets $\mathcal{A}$ and $\mathcal{A}_N$; (b) Ordered sets $\mathcal{P}_N$, $\mathcal{A}_N$ and the approximating attractor A^*.

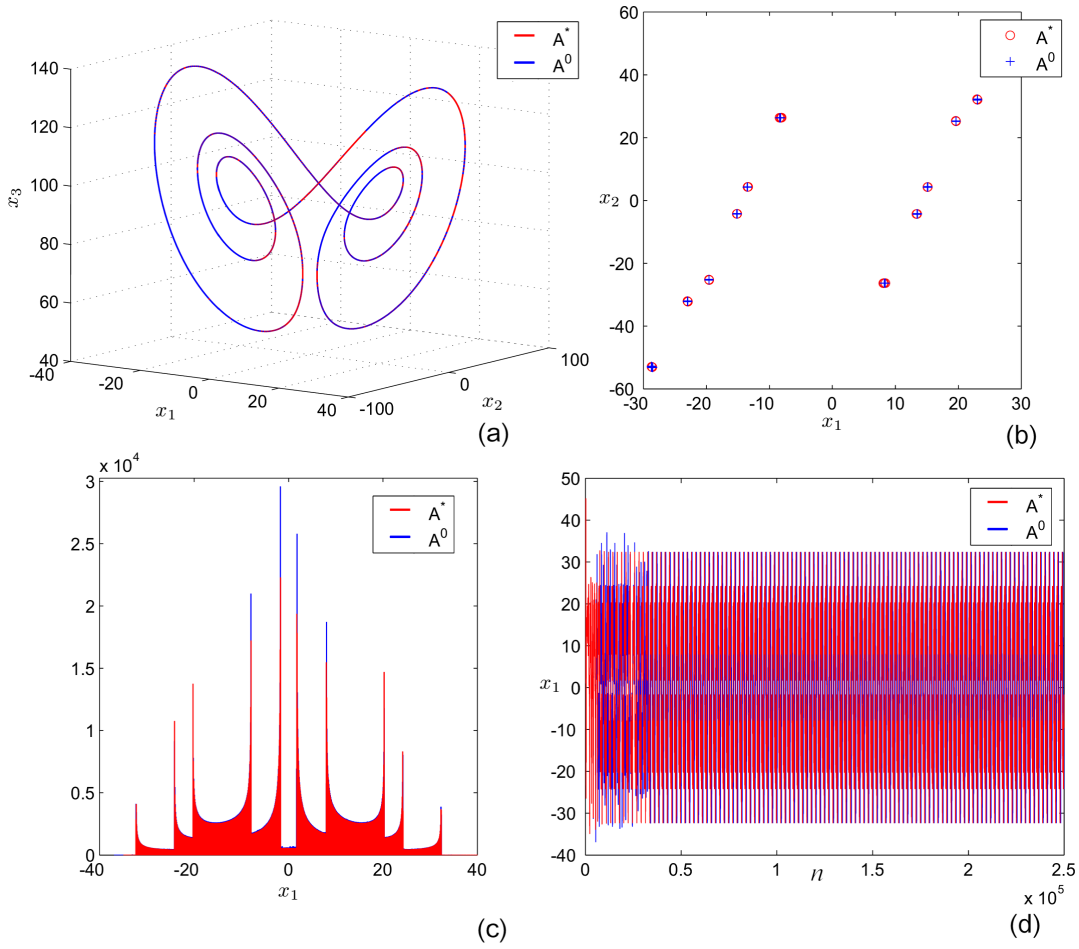


Figure 2: PS method $[m_1 \circ p_1, m_2 \circ p_2, m_3 \circ p_3]$, with $p_1 = 86$, $p_2 = 95$, $p_3 = 97$ and weights $m_1 = 2$, $m_2 = 1$, $m_3 = 3$ used to approximate the attractor A^0 corresponding to $p^0 = 93$. (a) Phase overplot of A^0 (blue) and A^* (red) ; (b) Overplot Poincaré sections with $x_3 = 100$; (c) Overplot histograms; (d) Overplot time series of the first component x_1.

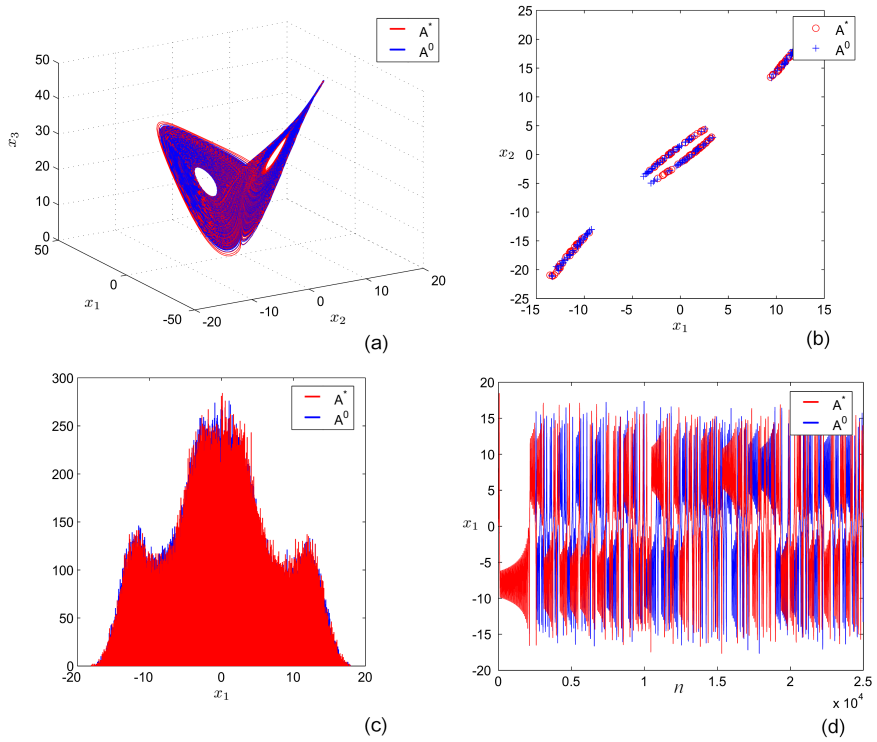
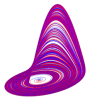
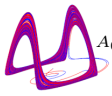


Figure 3: PS method $[m_1 \circ p_1, m_2 \circ p_2]$, with $p_1 = 24$, $p_2 = 28.5$, and weights $m_1 = 2$, $m_2 = 1$, used to approximate the chaotic attractor A^0 corresponding to $p^0 = 25.5$. (a) Phase overplot of A^0 (blue) and A^* (red); (b) Overplot Poincaré sections with $x_3 = 20$; (c) Overplot histograms;(d) Overplot time series of the first component x_1.

Table 1: Examples of attractors decomposition (switched attractor A^* and averaged attractor A^0 are plot in red and blue plot respectively).

System	Scheme	A^0 and A^*	A^0 Decomposition
Cancer system	$p^0 = 0.555, N = 5$ $\mathcal{P}_5 = \{0.198, 0.302, 0.402, 0.710, 0.806\}$ $\mathcal{M}_5 = \{2, 2, 1, 4, 3\}$ $\alpha_1 = \alpha_2 = \frac{1}{6}, \alpha_3 = \frac{1}{12}, \alpha_4 = \frac{1}{3}, \alpha_5 = \frac{1}{4}$		$A_{0.555} = \frac{1}{6} \otimes A_{0.198} \oplus \frac{1}{6} \otimes A_{0.302}$ $\oplus \frac{1}{12} \otimes A_{0.402} \oplus \frac{1}{3} \otimes A_{0.702} \oplus \frac{1}{2} \otimes A_{0.806}$
	$p^0 = 0.544, N = 2$ $\mathcal{P}_2 = \{0.535, 0.553\}$ $\mathcal{M}_2 = \{1, 1\}$ $\alpha_1 = \alpha_2 = \frac{1}{2}$		$A_{0.544} = \frac{1}{2} \otimes A_{0.535} \oplus \frac{1}{2} A_{0.553}$
Rössler system	$p^0 = 8.5, N = 4$ $\mathcal{P}_4 = \{8.28, 8.41, 8.540, 8.65\}$ $\mathcal{M}_4 = \{2, 1, 2, 3\}$ $\alpha_1 = \alpha_3 = \frac{1}{4}, \alpha_2 = \frac{1}{8}, \alpha_4 = \frac{3}{8}$		$A_{8.5} = \frac{1}{4} \otimes A_{8.28} \oplus \frac{1}{8} \otimes A_{8.41} \oplus \frac{1}{4} \otimes A_{8.54}$ $\oplus \frac{3}{8} \otimes A_{8.65}$
	$p^0 = 17, N = 2$ $\mathcal{P}_2 = \{17, 19\}$ $\mathcal{M}_2 = \{1, 1\}$ $\alpha_1 = \alpha_2 = \frac{1}{2},$		$A_{18} = \frac{1}{2} \otimes A_{17} \oplus \frac{1}{2} \otimes A_{19}$
RF system	$p^0 = 0.265, N = 3$ $\mathcal{P}_3 = \{0.2615, 0.2642, 0.2667\}$ $\mathcal{M}_3 = \{1, 2, 3\}$ $\alpha_1 = \frac{1}{6}, \alpha_2 = \frac{1}{3}, \alpha_3 = \frac{1}{2}$		$A_{0.265} = \frac{1}{6} \otimes A_{0.2615} \oplus \frac{1}{3} \otimes A_{0.2642}$ $\oplus \frac{1}{2} \otimes A_{0.2667}$
	$p^0 = 0.2715, N = 2$ $\mathcal{P}_2 = \{0.2714, 0.2716\}$ $\mathcal{M}_2 = \{1, 1\}$ $\alpha_1 = \alpha_2 = \frac{1}{2}$		$A_{0.2715} = \frac{1}{2} \otimes A_{0.2714} \oplus \frac{1}{2} \otimes A_{0.2716}$