Numerical approximations of a class of switch dynamical systems

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March 16, 2013

Abstract

The problem of numerically approximating a class of dynamical systems with discontinuous state variables by forward Euler method for differential inclusions, viewed as dynamical systems, is discussed in this paper. It is shown that such discontinuous initial value problems may be transformed into set-valued problems and then approximated by special numerical methods for differential inclusions which may be viewed as (ideally continuous) dynamical systems.

Key words: complex dynamics, differential inclusion, explicit Euler method, Filippov regularization, switching system.

1 Introduction

Consider the following initial value problem (i.v.p.) of a differential equation with discontinuous right-hand side:

$$\dot{x}(t) = f(x(t)) := g(x(t)) + \sum_{i=1}^{n} \alpha_i \operatorname{sgn} x_i(t)\, e^i,\ x(0) = x_0,\ t \in I = [0, \infty), \quad (1)$$

where $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are single-valued vector functions, $\alpha_i \in \mathbb{R}$ and e^i denote the ith canonical unit vectors in $\mathbb{R}^n$. The function f is assumed piecewise continuous, i.e., continuous on a finite number of open domains $D_i \subset \mathbb{R}$, $i = 1, .., p$, in each of which f is continuous up to a certain order, and has finite (possibly different) limits from different boundary points (i.e., bounded discontinuities).

The basic properties of this class of discontinuous dynamical systems, called switching systems, are now being intensively studied, despite the tedious work necessary for investigating the underlying i.v.p. Generally, many physical laws are expressed by this kind of discontinuity and occur in real-world applications such as the discontinuous dependence of friction force on velocity in the case of dry friction, oscillating systems with combined dry and viscous damping effects, chaotic circuits, alternatively forced vibrations, braking processes with locking phases, convex optimizations, non-smooth control systems synthesis, uncertain systems, etc. (see e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11], and some references therein).

One way to deal with the model (1) could be to apply an approximation of the discontinuous functions f, in the proximity of the discontinuities, with suitable continuous functions. In doing so, the i.v.p. becomes a continuous one [12] and the many available tools for continuous and smooth dynamical systems can be employed. Another approach could be to use numerical approximations of the discontinuous problems.

In fact, for the case of a continuous dynamical system, numerical approximation of the underlying system is well known (see e.g. [13]), which will be followed in this paper. More precisely, the approach of numerically approximating the discontinuous dynamical system, modeled as the i.v.p. by (1), is studied. This problem is important, since almost all tools for dynamical systems employ numerical methods (see e.g. [14, 15, 16, 17]). In other words, the basic properties of the dynamical system considered in this paper will be analyzed based on its numerical approximation. Due to the discontinuity of the right-hand side, the i.v.p. (1) may not have any solution, therefore a different concept of "solution" must be defined and used. For this purpose, the i.v.p. (1) will be transformed into a differential inclusion problem via the well-known Filippov regularization [18]. Having enough regularity, the new i.v.p. so obtained may have several generalized solutions, which can be computed by numerical methods.

In this paper, the classical explicit Euler method will be employed for differential inclusions, under the assumption that the switching system modeled by (1) can be numerically well-approximated using the explicit Euler method. This could be considered as a discrete dynamical system that well approximates the original switching dynamical system.

The paper is organized as follows. Section 2 presents some assumptions on f under which the i.v.p. (1) defines a switching dyanmical system. Section 3 reviews the explicit Euler method and Section 4 discusses the assumptions on f under which the switching dyanmical system (1) can be well-approximated by the Euler method. Section 5 concludes the paper. Throughout, three examples are given with simulations.

2 Switching dynamical systems

In order to define the class of dynamical systems modeled by the i.v.p. (1), a few notions are first introduced, with some preliminary results.

One of the basic assumptions is the so-called *growth condition*, which is used here instead of requiring the global boundedness of the right-hand side [19, 20].

Definition 1 *A set-valued function F is said to satisfy a* growth condition*, if* *there exist constants* $K_1, K_2 \geq 0$ *such that*

$$\| \xi \| \leq K_1 \| x \| + K_2, \tag{2}$$

for all $\xi \in F(x),\ x \in \mathbb{R}^n$.

Due to the discontinuity of the right-hand side, the i.v.p. (1) may not have any solution. For example, consider the following discontinuous problem

$$\dot{x} = 2 - 3\, sgn(x), \tag{3}$$

which has solutions

$$x(t) = \begin{cases} 5\,t + C_1, & x < 0 \\ -t + C_2, & x > 0. \end{cases}$$

As t increases, these classical solutions tend to the line $x = 0$, but they cannot continue to evolve along this line since the function $x(t) = 0$ does not satisfy the equation. Thus, there is no classical solution starting from 0.

To introduce a sensible solution to the above i.v.p., Filippov [18] proposed the idea of restarting the i.v.p. as the following differential inclusion:

$$x(t) \in F(x(t)),\ x(0) = x_0, \quad for \quad a.a. \ \ t \in I, \tag{4}$$

where $F : \mathbb{R}^n \Longrightarrow \mathbb{R}^n$ is a set-valued vector function defined on the set of all subsets of $\mathbb{R}^n$. The simplest definition of F is

$$F(x) = \bigcap_{\varepsilon > 0} \bigcap_{\mu(M)=0} \overline{conv(f\{z \in \mathbb{R}^n : \| x \| \leq \varepsilon\} \backslash M))}, \tag{5}$$

where μ is the Lebesgue measure, M is the set of all discontinuity points, $conv(\cdot)$ is the closed convex hull, and f is a single-valued function which is discontinuous with respect to the state variable. At the points where the function f is continuous, $F(x)$ consists of one point, which coincides with the value of f at this point. At the discontinuity points, the set $F(x)$ is a subset of $\mathbb{R}^n$ given by (5).

In order to justify the use of the Filippov regularization in physical systems, ε must be small enough, so that the motion of the physical system can be arbitrarily close to a certain solution of the differential inclusion. As an example, consider the set-valued version of the usual a *sign* function, i.e., the set-valued Sgn function defined by

$$Sgn(x) = \begin{cases} \{-1\}, & x < 0 \\ [-1, 1], & x = 0 \\ \{+1\}, & x > 0. \end{cases}$$

Using Filippov regularization, the i.v.p. (1) becomes

$$l\dot{x}(t) \in F(x(t)) := g(x(t)) + \sum_{i=1}^{n} \alpha_i\, Sgn\, x_i(t)\, e^i,$$
$$x(0) = x_0, \quad for \quad a.a.\ t \in I = [0, \infty). \tag{6}$$

Notation 2 *Let L denote the class of functions f defined by (1) with Lipschitz continuous g, for which the corresponding set-valued form F satisfies a growth condition.*

Definition 3 *[18] A* generalized (Filippov) solution *to the i.v.p.* (1) *is an absolutely continuous function, $x : I \longrightarrow \mathbb{R}^n$, satisfying (6) a.e. on I.*

Basic properties of Filippov solutions and background of differential inclusions can be referred to, e.g., [19, 20]. Under Filippov regularization, the i.v.p. (1) may have several generalized (Filippov) solutions.

Definition 4 *The i.v.p.* (1) *is said to define a generalized switching dynamical system on $\mathbb{R}^n$ if, for every $x_0 \in \mathbb{R}^n$, there exists a solution of the i.v.p.* (1) *defined for a.a. $t \in I$. Furthermore, if the solution is a.e. unique, then the i.v.p.* (1) *is said to define a switching dynamical system.*

A condition on f under which the i.v.p. (1) defines a switching dynamical system is given by the following theorem.

Theorem 5 *[21] The i.v.p.* (1) *with $f \in L$ defines a generalized switching dynamical system. If, moreover, all the coefficients α are nonnegative, then the i.v.p.* (1) *defines a switching dynamical system.*

Proof. In [21], it is proved that if $f \in L$, the i.v.p. (1) has at least one generalized solution, while if all the coefficients α are non-positive, then the i.v.p. (1) has a unique solution. ■

Using the above result, the transformed i.v.p. (3) becomes

$$\dot{x} \in 2 - 3\ Sgn(x).$$

Since the right-hand side belongs to class L, the system has a unique positive generalized solution defining a switching dynamical system: if $x_0 > 0$ then $x(t) = -t + x_0$ for $t < x_0$ and $x(t) = 0$ for $t \geq x_0$; namely, the solution can be continuously prolonged from 0. Also, there is a unique negative solution for $x'_0 < 0$: $x(t) = 5t + x'_0$ for $t < x'_0$, and $x(t) = 0$ for $t \geq x'_0$ (Figure 5).

Figure 1

Next, consider the following discontinuous model, which is a generalization of Chua's circuit ([21, 22])

$$\begin{aligned} \dot{x}_1 &= -2.57x_1 + 9x_2 + sgn(x_1) \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -\beta\, x_2, \end{aligned} \tag{7}$$

where β is the control parameter. It is easy to check that (7) models a generalized switching dynamical system, since the positiveness of the α coefficients and the Lipschitz continuity of the g function (Theorem 5)

$$g(x) = (-2.57x_1 + 9x_2,\ x_1 - x_2 + x_3,\ -\beta\ x_2\ )^T.$$

The following system is a simplified model of the regulation system of a steam turbine [23]

$$\begin{array}{rcl} \dot{x}_1 & = & x_3 - x_1 - sgn(x_2) \\ \dot{x}_2 & = & x_1 - x_2 \\ \dot{x}_3 & = & -x_2, \end{array} \tag{8}$$

which defines a switching dynamical system.

A modified mathematical description of the Chen system [24] has the following form

$$\begin{array}{rcl} \dot{x}_1 & = & a(x_2 - x_1) - 0.5\, sgn(x_1) \\ \dot{x}_2 & = & x_1(c - a - x_3) + cd\, x_2 + 0.5\, sgn(x_2) \\ \dot{x}_3 & = & x_1 x_2 - b\, x_3 + 3\, sgn(x_1), \quad (a, b, c, d > 0), \end{array} \tag{9}$$

which defines a generalized switching dynamical system, since not all α coefficients are negative. The control parameter is c, with other parameters $a = 1.18, d = 0.1, b = 0.168$.

3 Explicit Euler method for differential inclusions

In order to numerically solve the i.v.p. (1), some effective numerical methods for solving the i.v.p. (6) are needed. In the following, the simplest set-valued version of the explicit Euler method for differential equations is reviewed.

Consider the general case of a non-autonomous differential inclusion

$$\dot{x}(t) \in F(t, x(t)),\ x(t_0) = x_0, \quad for \quad a.a. \ \ t \in [t_0, T]. \tag{10}$$

This method consists of a replacement of the original difference inclusion on the interval $[t_0, T]$ by a sequence of discrete inclusions on a sequence of grids (see [25, 26, 27, 28, 29, 30] and some references therein)

$$t_0 = t_0^N < t_1^N < \cdots < t_N^N = T,$$

with step-size

$$h = \frac{T - t_0}{N}, \quad (N \in \mathbf{N'} \subset \mathbb{N}),$$

and in the subsequent numerical solutions of these discrete inclusions. This results in a sequence

$$\left(\eta^N\right)_{N\ \in \mathbf{N'}},$$

of grid functions

$$\eta^N = \left(\eta_0^N,\ \eta_1^N, \cdots, \eta_N^N\right), \quad N \ \in \mathbf{N}'.$$

Here, $\mathbf{N}'$ denotes a subsequence of $\mathbb{N}$ converging to infinity. For simplicity, N is use as the index. The simplest explicit digitization method for differential inclusions is the following explicit Euler method

$$\eta_{k+1} = \eta_k + h\,\xi_k, \quad \xi_k \in F(t_k, \eta_k), \quad k = 0, 1, \cdots, N-1, \quad \eta_0 = x_0. \tag{11}$$

The convergence of this Euler method, when being applied to the i.v.p. (6) with $T = \infty$ for $f \in L$, can be found in [21]. Generally, the solution of the inclusion

$$\xi_k \in F(t_k, \eta_k)$$

is not unique. So, it is usually randomly selected, as in the present paper, or by a suitable optimization (see e.g. [30]). As an example, consider the generalized Chua switching dynamical system (7). The corresponding set-valued i.v.p. of the differential inclusion is

$$\begin{array}{rcl} \dot{x}_1 & \in & -2.57x_1 + 9x_2 + Sgn(x_1) \\ \dot{x}_2 & = & x_1 - x_2 + x_3 \\ \dot{x}_3 & = & -\beta\, x_2. \end{array}$$

A chaotic trajectory is plotted in Figure 5. For the switching dynamical system modeled by (8), a stable trajectory is drawn in Figure 5, while a chaotic trajectory of the system (9), with $c = 9.7$, is plotted in Figure 5 and 5. These phase portraits and time series were plotted by using the explicit Euler method. The trajectories present some corners, typical for first-order convergence methods (like the explicit Euler method) when they cross the discontinuity surfaces.

Figures 2,3,4,5

4 Numerical approximation of the i.v.p. (1.1)

Consider a sequence $\{y_n\}_{n\in N}$ satisfying

$$y_{n+1} \in G(y_{n+1}), \quad y(0) = y_0 \in \mathbb{R}^n, \tag{12}$$

where $G : \mathbb{R}^n \Longrightarrow \mathbb{R}^n$ is a set-valued map.

Using definition 3 and 4, one can consider that the set-valued i.v.p. (12) defines a discrete dynamical system.

Definition 6 *The i.v.p.* (12) *is said to define a generalized dynamical system on $\mathbb{R}^n$ if, for every $y_0 \in \mathbb{R}^n$, there is a solution defined for all $n \geq 0$. Furthermore, if this solution is unique, then the i.v.p.* (12) *is said to define a dynamical system.*

If the sequence $\{y_n\}_{n\in N}$ is generated by some numerical methods for solving the i.v.p. (6) and is convergent, it is said that the underlying discontinuous i.v.p. (1) is numerically approximated by the i.v.p. (12).

For a given step-size h, the discrete i.v.p. (12) can be transformed into an ideally continuous one by using the following continuous piece-wise linear function

$$y : I \longrightarrow \mathbb{R}^n,$$

so that in each interval $[t_j, t_{j+1}]$,

$$y(t) = y_j + \frac{1}{h}(t - t_j)(y_{j+1} - y_j), \quad t_j \leq t \leq t_{j+1}, \quad j = 0, 1, \cdots, N-1.$$

The existence of solutions of (12) is always guaranteed, while the uniqueness has to be checked. Nevertheless, (12) always gives at least one generalized dynamical system solution.

Now, consider the i.v.p. (1) with the corresponding differential inclusion (6). If we want to numerically approximate the i.v.p. with a discrete dynamical system via some numerical method, e.g. the explicit Euler method, we have to consider the condition on f, under which the numerical scheme (11) applied to (6) has at least one solution. Using Theorem 5, we obtain the following main result of this paper.

Theorem 7 *Consider the i.v.p.* (1) *with* $f \in L$. *In this case, the explicit Euler method* (11) *defines a generalized dynamical system, which numerically approximates the underlying dynamical system. Moreover, if all* α *coefficients are non-positive, then* (11) *defines a switching dynamical system.*

Remark 8 *In the case of numerically approximating a continuous dynamical system, there exists a system whose i.v.p. does not define a dynamical system, even if the Euler method defines a dynamical system (e.g. the i.v.p.* $\dot{x} = x^2,\ x(0) = x_0$, *whose solution exists only for* $t \neq 1/x_0$ *).*

As an example, consider the i.v.p. (7). The forward Euler numerical approximation represents a discrete dynamical system, which is

$$\begin{aligned} x_{1n+1} &\in x_{1n} - 2.57x_{1n} + 9x_{2n} + Sgn(x_{1n}) \\ &\quad = -1.57x_{1n} + 9x_{2n} + Sgn(x_{1n}) \\ x_{2n+1} &= x_{2n} + x_{1n} - x_{2n} + x_{3n} = x_{1n} + x_{3n} \\ x_{3n+1} &= x_{3n} - \beta x_{2n}. \end{aligned}$$

5 Conclusions

In this paper, we have proved that the class of switching dynamical systems modeled by (1) with $f \in L$ can be numerically approximated by the explicit Euler method.

There are a few questions remaining to be answered: 1) assuming that the differential inclusion (6) admits invariant sets (such as steady-state solutions, periodic solutions, quasi-periodic solutions, chaotic solutions, basins of attraction), does a numerical method for differential inclusions possess the corresponding invariant sets? 2) assuming that the vector field defining the differential inclusion has a particular structural property, what are the numerical methods for differential inclusions that can inherit these structural properties? 3) what are the conditions under which numerical orbits represent the true orbits (shadowing theory; see [31] for the continuous case)?

Moreover, there are some interesting results that could be obtained, such as synchronization of several chaotic switching dynamical systems [32], control of discontinuous dynamical systems [33], and chaotification of stable trajectories of a switching dynamical system [34].

Finally, further analyzing various numerical approximation methods and applied them to more general non-smooth i.v.p. remains a challenging task for future research.

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Figures caption

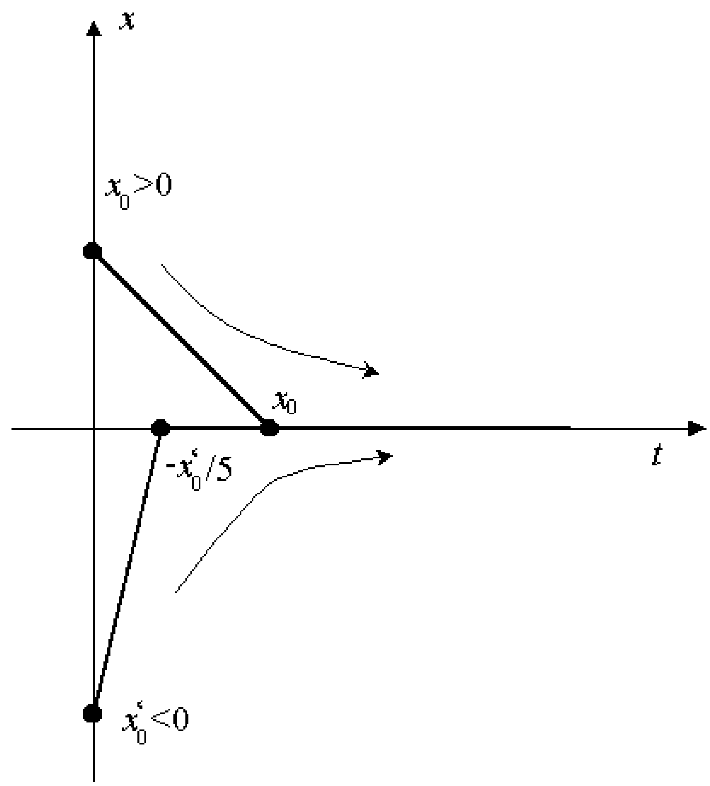
Figure 1. Two trajectories of the initial value problem (3).

Figure 2. A chaotic trajectory of the system (7) (three-dimensional plot).

Figure 3. A stable trajectory of the switching dynamical system (8) (time series and phase portraits).

Figure 4. A chaotic trajectory of the switching dynamical system (9) (three-dimensional plot)

Figure 5. The same trajectory as in Figure 4 (time series and phase portraits)



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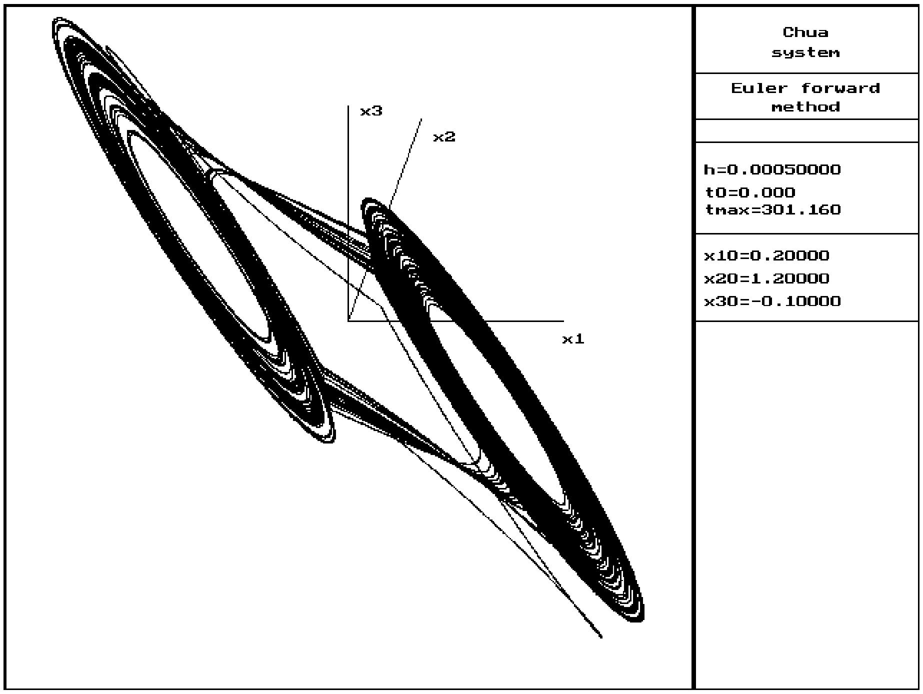
x0>0
x0
t
-x'0/5
x'0<0


Chua
system
Euler forward
method
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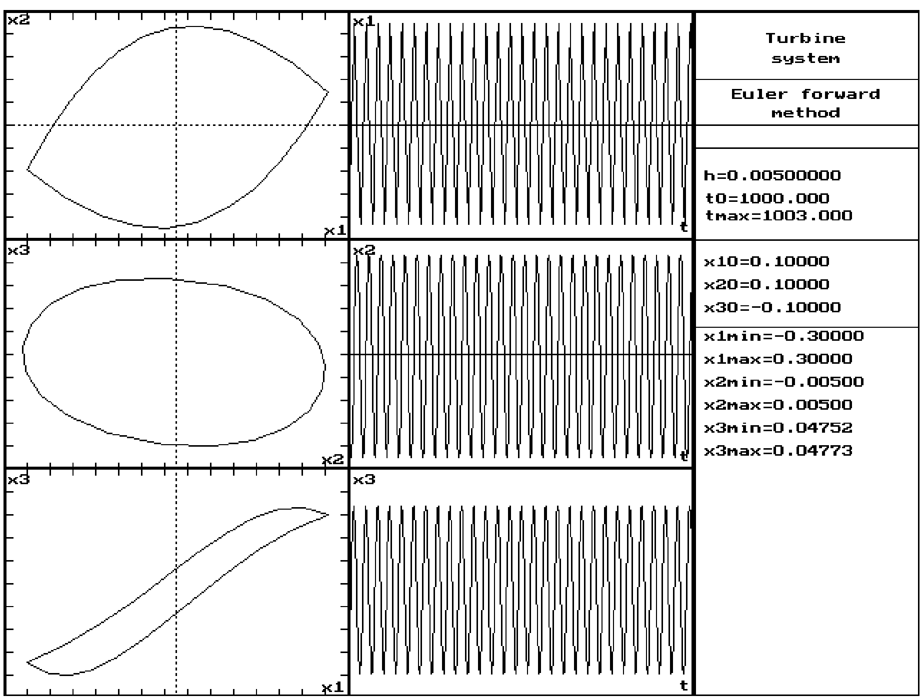
x20=1.20000
x30=-0.10000
x3
x2
x1


Turbine
system
Euler forward
method
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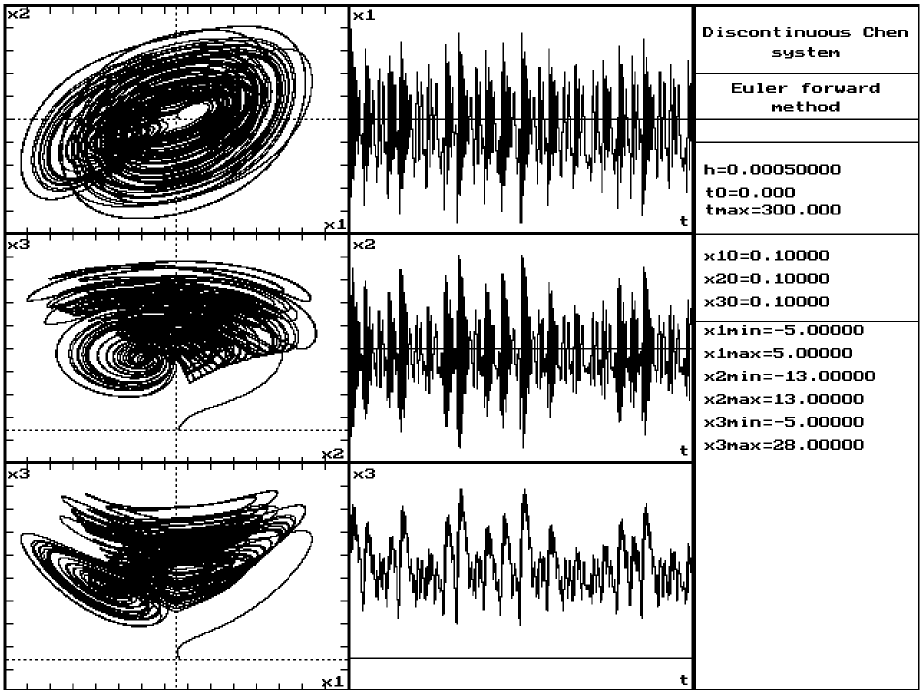
tmax=1003.000
x10=0.10000
x20=0.10000
x30=-0.10000
x1min=-0.30000
x1max=0.30000
x2min=-0.00500
x2max=0.00500
x3min=0.04752
x3max=0.04773
x2
x1
x3
x2
x3
x1
x1
t
x2
t
x3
t


x2
x1
t
x3
x2
t
x3
x1
t
Discontinuous Chen system
Euler forward method
h=0.00050000
t0=0.000
tmax=300.000
x10=0.10000
x20=0.10000
x30=0.10000
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x1max=5.00000
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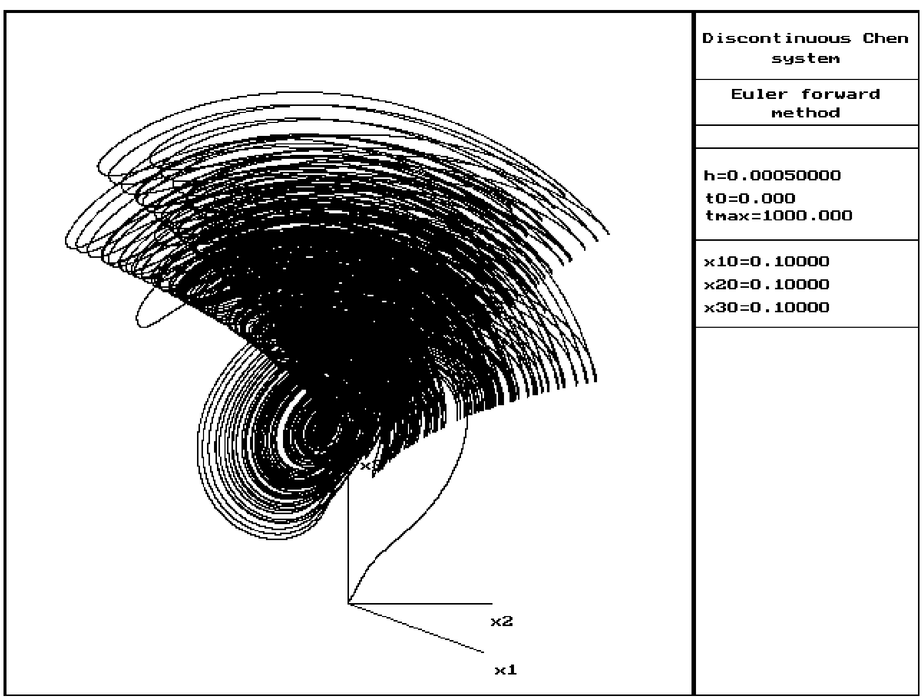
x2
x1
Discontinuous Chen system
Euler forward method
h=0.00050000
t0=0.000
tmax=1000.000
x10=0.10000
x20=0.10000
x30=0.10000