Chaos suppression via periodic change of variables in a class of discontinuous dynamical systems of fractional order

Marius-F. Danca

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Abstract By introducing a suitable change of variable theorem for a class of fractional discontinuous equations, we study to possibility to use a periodical perturbation algorithm to stabilize chaotic trajectories. For this purpose, some new issues of fractional differential inclusions and results on Filippov systems are used. The algorithm, which change periodically the system variables, has been used so far to stabilize discrete, continuous and discontinuous systems of integer order. As example, a piece-wise continuous variant of the Chen system is utilized.

Keywords fractional differential equations, fractional differential inclusions, piece-wise continuous systems, chaos control

1 Introduction

Fractional differential equations have become recently an object of increasing interest being widely implied in several scientific domains such as engineering, biology, medicine finance, in many interdisciplinary fields, in purely mathematical or heuristic basis etc. and proved valuable tools to model physical phenomena. There are many real systems which can be described by fractional differential equations much more efficiently than by classical techniques (a complete bibliography on applied problems which require definitions of fractional derivative can be found in [1] and [2]). The applied problems require definitions of fractional derivatives physically interpretable for initial conditions, one of them being Caputo's fractional derivative, introduced in [3] (for other definitions for fractional time derivatives we refer to [4–6]).

The main reason for using only integer-order models was the absence of solution methods for fractional differential equations and the multiple nonequivalent definitions of fractional derivatives. Therefore, during the last 10 years, a significant theoretical development in fractional differential equations has been done; see e.g. the consistent monographs of Podlubny [7], Kilbas et al [8], Samko et al [6] Kai [9] Miller and Ross [4] or the papers of Kilbas et al [10], Podlubny [5], Kai et al [11,12] Kilbas and Trujillo [10], Momami et al [13], Yu and Gao [14] Benchohra et al [15] Tavazoei [16].

During the last ten years, fractional differential inclusions, initially studied by El-Sayed and Ibrahim [17], have been intensely studied by many mathematicians (see e.g. the books of Aubin [18], Benchohra et al [19], Benchohra et al [19], or papers [1,2,20]).

One of the first discontinuous model of the friction contact between solid bodies, was introduced by Coulomb model circa 1781 [21]. Sine then, the Coulomb model is widely used, being accurate enough in many engineering applications. However it gives rise to difficult computational and analytical problems. Many processes in industry and elsewhere, exhibit switches regime, which may be described as fast phenomena which may lead to large changes in the system dynamics and the system state. Such switches may be due to external causes (perturbations) or may occur as a result of the process dynamics itself. The examples include robotic mechanisms which switch from compliant to non-compliant modes, electrical networks in which for instance thyris-

Marius-F. Danca
Dept. of Mathematics and Computer Science
Avram Iancu University
400380, Cluj-Napoca, Romania

and
Romanian Institute of Science and Technology,
400487 Cluj-Napoca, Romania
E-mail: danca@rist.ro

tors switch from conducting to non-conducting mode, processes under the influence of a discrete (for instance on/off) controller, thermostats implement closed-loop bang-bang controllers to regulate room temperature, aerial and underwater terrains are examples where discontinuities naturally occur from the interaction with the environment and the list can go on. Examples and background theory on this field can be found in early works like [22,23], or in recent works such as [24,25] or [26].

Therefore, taking account of the above synthesis, it is easy to understand why discontinuous systems modeled by fractional discontinuous differential equations represent a very recent and promising subject.

In this paper we show that an extension of a Filippov's result [27] for a class of discontinuous systems can be adapted via fractional differential inclusions, to a class of discontinuous systems of fractional order. In this way, the use of a perturbation algorithm used to control continuous nonlinear systems of integer order, presented in several papers such as [28–31] and called hereafter Variable Perturbation Algorithm (VPA), is allowed to our class of systems.

The analyzed class of systems are modeled by the following Initial Value Problem (IVP)

$$D_*^q x(t) = f(x(t)), \quad x(0) = x_0, \quad t \in I = [0,T], \quad T > 0, \tag{1}$$

where $f : E \subseteqq \mathbb{R}^n \to \mathbb{R}^n$ is a piece-wise continuous function. As a common practice for many applications, D_*^q stands for the Caputo differential operator with starting point 0 and $0 < q < 1$, which allows the used of initial consitions in the standard form, and have the following form ([3,32])

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-\tau)^{-q} u'(\tau)\, d\tau.$$

Γ is the Gamma function or Euler's integral of the second kind, defined by

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx,$$

which, for integer z, is

$$\Gamma(z+1) = z!$$

There are several ways to approximate Gamma function Γ, one of the most utilized being the Lanczos approximation [33].

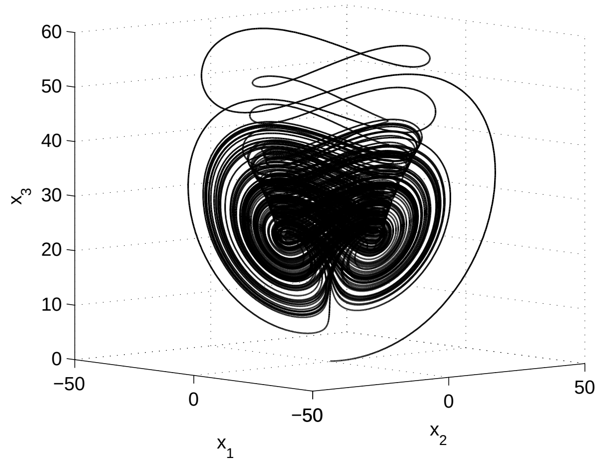

Fig. 1 Chaotic attractor of Chen system (4).

The paper is organized as follows: in Section 2 we present VPA for the simple case of continuous systems. In Section 3 we recall some notions and results on fractional set-valued initial valued problems. In Section 4 we introduce the main result: the change of variable for a class of fractional discontinuous differential equations. In Section 5 the result is applied for a particular class of fractional discontinuous systems and an example is presented. In the last section some open problems and future directions are pointed out.

2 VPA for continuous systems of integer order

Let consider the continuous autonomous IVP of integer (first) order

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \in I, \tag{2}$$

where f is a nonlinear function defined and continuous on some subset E of $\mathbb{R}^n$, and $x_0 \in E$.

Used generally as a chaos suppression algorithm, VPA implies pulses applied to perturb sporadically the state variable x in (2) [29,30]

$$x(t+) = [1 + \lambda\delta(t - j\Delta t)]x(t-), \quad j = 0, 1, 2, ... \tag{3}$$

where λ, a small (positive or negative) real number, regulates the intensity of the perturbation, via the Dirac function δ, at each $j\Delta t$ and $x(t\pm) = \lim_{h\to 0} x(t \pm h)$ (see also the approach in [28,34,35] for discrete systems and [36] for discontinuous of integer order systems). The simplest numerical implementation of this algorithm is to use some scheme for ODEs with fixed step size h, and apply periodically the perturbations (changes of variables) (3) after every time interval $\Delta t = mh$, for some positive integer m, which computationally means

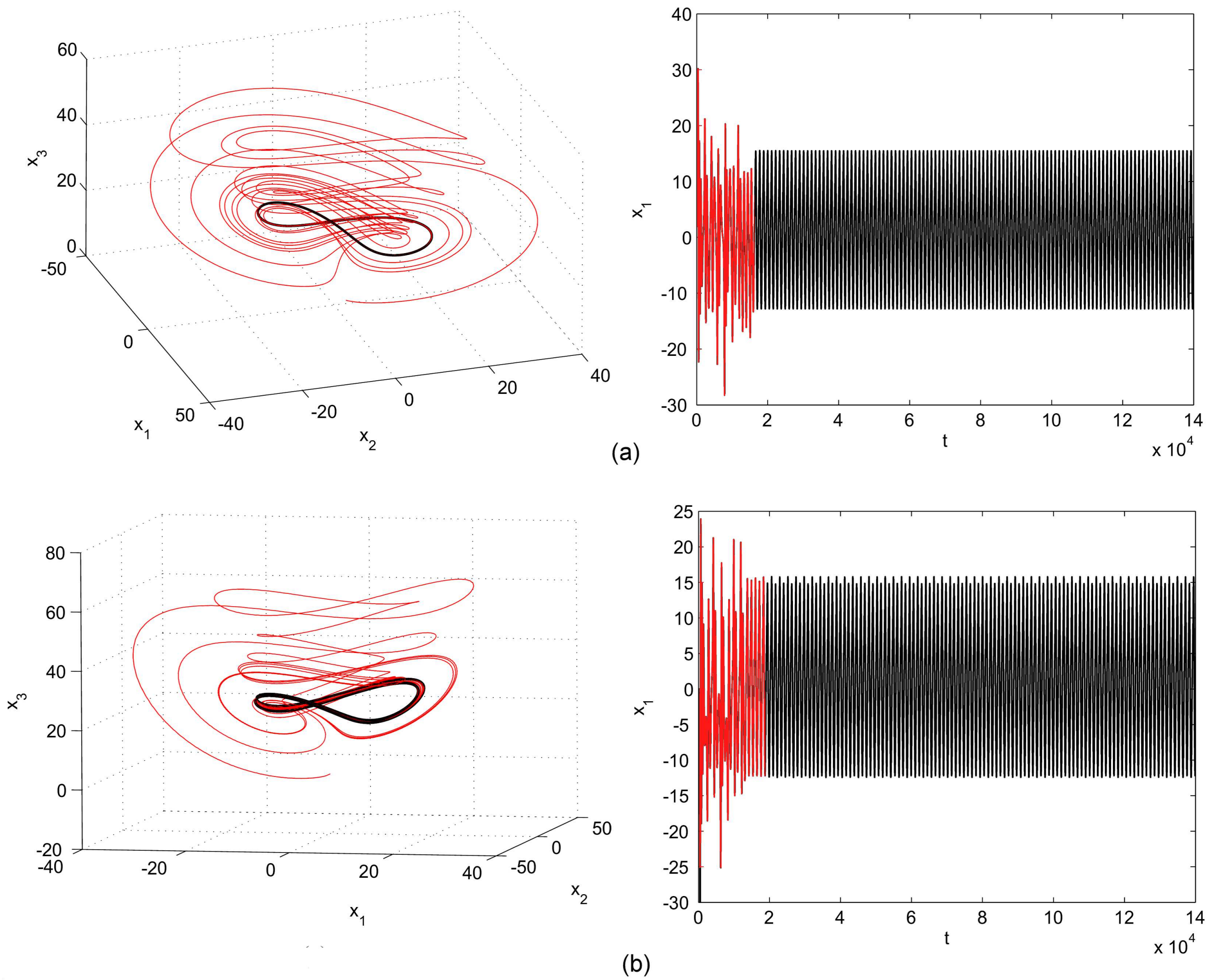

Fig. 2 Phase plots and time series for stable limit cycles of the Chen system (4) obtained with VPA (red plot for transients). (a) $\Delta t = 0.001$ sec and $\lambda = -0.009$. (b) $\Delta t = 0.001$ sec and $\lambda = -0.009$.

$$x_+ \leftarrow (1+\lambda)x_-, \quad \text{every} \quad j\Delta t, \quad j = 0, 1, 2, ...$$

where x_- is the value of the state variable after $j\Delta t$ integration steps and x_+ is the new value at the moment $j\Delta t$.
For example, let us consider the Chen system [37]

$$\begin{aligned} \dot{x_1} &= a(x_2 - x_1), \\ \dot{x_2} &= cx_2 + (c - a - x_3)x_1, \\ \dot{x_3} &= -bx_3 + x_1x_2, \quad a, b, c \in \mathbb{R}^+, \end{aligned} \tag{4}$$

which, for $a = 35, b = 3$ and $c = 26$, evolves chaotic (Figure 1). For this parameter values, if one apply VPA every $\Delta t = 0.001$ sec, with $\lambda = -0.009$, the system will evolve on a stable limit cycle (Figure 2 a), while for $\Delta t = 0.003$ sec and $\lambda = -0.02$, one obtains the stable limit cycle plotted in Figures 2 b. As can be seen, there are a transient time interval until the trajectory reaches the stable behavior.

Remark 1
(i) Different λ values for each variable can be used to obtain better results [31];
(ii) An alternative form of the proportional like perturbation (3) is to use additive pulses: $x(t+) = x(t-) + \lambda\delta(t - j\Delta t)$ [31];

Even, to our best knowledge, there exist only empirical rules to choose the parameters in (3), compared to the classical control algorithms, where tedious calculations are necessary, the simplicity of VPA recommends it as an useful chaos control method since it do not require any knowledge about the model equations or the system dynamics. Moreover, this way could be 'responsible' for some unexpected controlled behaviors in real or natural systems.

Therefore, we are motivated to implement it for the class of discontinuous systems of fractional order modeled by IVP (1).

3 Preliminaries

In this section we recall some known definitions, and results.

Let X and Y be metric spaces and a set-valued function F from X to Y. If the images of F are closed, convex, bounded, and so on, we say that F is closed-valued, convex-valued, bounded-valued, and so on. F is upper semicontinuous (u.s.c.) on X if for each $x^0 \in X$ and for any open set N containing $F(x^0)$ there exists a neighborhood M of x^0 such that $F(M) \subset N$ (see e.g. [38] or [39]).

We let $AC^1(I, \mathbb{R}^n)$ the space of differentiable functions whose first derivative is absolutely continuous.

$L^1(I, \mathbb{R}^n)$ denotes the Banach space of functions $x : I \to \mathbb{R}^n$ that are Lebesgue integrable with the norm

$$||x||_{L^1} = \int_0^T |x(t)|dt.$$

Let us introduce the following hypothesis

(H1) *f in the IVP (1) is piece-wise continuous on $E \subseteq \mathbb{R}^n$ with bounded discontinuities and M the null set of the discontinuities.*

As known, discontinuous IVPs of integer or fractional order may have not any classical solutions. To encompass this impediment for the case of discontinuous equations of integer order, Filippov gave an elegant solution [27]: he transformed, by so called *Filippov regularization*, the initial problem into a set-valued one, by embedding the discontinuous function f into a set-valued function F. If we use this way to our IVP (1) one obtains the following set-valued IVP of fractional order

$$D_*^q x(t) \in F(x(t)), \quad x(0) = x_0, \quad \text{a.e. } t \in I, \tag{5}$$

where $F : E \Longrightarrow \mathcal{P}(E)$ is a set-valued function, $\mathcal{P}(E)$ being the family of all nonempty subsets of $E \subseteq \mathbb{R}^n$ and $q \in (0, 1)$. One of the most utilized definitions for F is the following [27]

$$F(x(t)) = \bigcap_{\varepsilon>0} \bigcap_{\mu(M)=0} \overline{conv} f\left((x(t) + \varepsilon B) \setminus M\right), \tag{6}$$

with B the unit ball in $\mathbb{R}^n$, and μ the Lebesgue measure. For the points where the function f is continuous, $F(x)$ will consist of one point, which is the value of f at this point, while at the discontinuity points, $F(x)$ is the convex hull of the values $f(x)$ given by (6), for $x \in M$. For example, the Filippov regularization applied to $sgn(x)$, for the particular case $n = 1$, leads to the multi-valued (sigmoid) function

$$Sgn(x) = \begin{cases} \{-1\} & \text{for} \quad x < 0 \\ [-1, 1] & \text{for} \quad x = 0 \\ \{+1\} & \text{for} \quad x > 0 \end{cases},$$

As shown in [27], F obtained with (6) enjoys the following properties: the set $F(x)$ is nonempty and F is u.s.c., bounded-valued, closed-valued and convex-valued (see also [40]).

Definition 1 $x \in AC^1(I, \mathbb{R}^n)$ is said to be a solution to IVP (5) if there exists $y \in L^1(I, \mathbb{R}^n)$ with $y(t) \in F(x(t))$ for a.e. $t \in I$ such that x satisfies the fractional differential equation $D_*^q x(t) = y(t)$ a.e. on I and verifies the condition $x(0) = x_0$.

Filippov's results, like the following theorem, topological structure of the solutions set and existence for fractional differential inclusions (with or without impulses) with convex as well as nonconvex valued right hand side, have been considered during the last years by many authors (see e.g. the papers by Henderson and Ouahabb [1] Benchohra et al [15,19], A. Cernea [41] and others)

Theorem 1 *Let the IVP (5) and (H1) holds. Then the IVP (5) has at least one solution.*

4 Change of variables in IVP (1)

The key of the proof of our main result is the following theorem concerning the change of variables applicable for discontinuous differential equations of integer order

Theorem 2 [27, Ch. 2, p. 101] *Suppose (H1) is satisfied and consider a continuous positive-valued function $p : E \mapsto \mathbb{R}^n$. Then the equations $x'(t) = p(x(t))f(x(t))$ and $x'(t) = f(x(t))$ have the same solutions in E.*

Following a similar way introduced by Filippov in [27, Ch. 1, p. 85] the following notion of solution to (1) can be introduced

Definition 2 $x \in AC^1(I, \mathbb{R}^n)$ is said to be a (generalized) solution to IVP (1) if it is a solution to IVP (5).

Now, we can introduce our main result which is a generalization of Theorem 2

Theorem 3 (**Change of Variable Theorem**) *Suppose (H1) is satisfied and let the continuous positive-valued function* $p : E \mapsto \mathbb{R}^n$. *Then the equations*

$$D_*^q x(t) = f(x(t)), \tag{7a}$$

$$D_*^q x(t) = p(x(t))f(x(t)), \tag{7b}$$

have the same solutions in E.

Proof Since we need the solutions to (7a) and (7b), we have to restart these equations as differential inclusions for t a.e. in I, to which Theorem 1 applies

$$D_*^q x(t) \in F(x(t)), \quad a.e. \ \in I, \tag{8a}$$

$$D_*^q x(t) \in p(x(t))F(x(t)), \quad a.e. \ \in I. \tag{8b}$$

Let $x(t)$ a solution of the inclusion (8b), that is $x(t)$ is verifies (8b) a.e. on I

$$D_*^q x(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \frac{dx(s)}{ds} \,\mathrm{d}s$$

$$:= v(t) \in p(x(t))F(x(t)).$$

On I denote

$$\tau(t) = \int_0^t p(x(s)) \,\mathrm{d}s. \tag{9}$$

Because $\tau'(t) = p(x(t)) > 0$ is continuous, $\tau(t)$ admits an inverse function $t(\tau)$.
The function $x^*(\tau) = x(t(\tau))$ is absolutely continuous [42] and

$$\begin{aligned} D_*^q x^*(\tau) &= D_*^q x(t(\tau)) \\ &= \frac{1}{\Gamma(1-q)} \int_0^\tau (\tau-s)^{-q} \frac{dx(t(s))}{ds} \,\mathrm{d}s \\ &= \frac{1}{\Gamma(1-q)} \int_0^\tau (\tau-s)^{-q} \frac{dx(t(s))}{dt} \frac{dt(s)}{ds} \,\mathrm{d}s \\ &= \frac{1}{p(x^*(\tau))} \frac{1}{\Gamma(1-q)} \int_0^\tau (\tau-s)^{-q} \frac{dx(t(s))}{dt} \frac{1}{p(x(t(s)))} \,\mathrm{d}s \\ &= \frac{v(t(\tau))}{p(x^*(\tau))} \in F(x^*(\tau)), \end{aligned}$$

for a.e. $\tau \in [0, T]$. Since the functions $\tau(t)'$ and $t'(\tau)$ are continuous, "for almost all t" and " for almost all τ" are equivalent [42]. Therefore, any solution of the inclusion (8a) is also solution of the differential inclusion (8b), and reversely, since the function $1/p(x) > 0$ is also continuous.

Next, since the solution of the differential equations (7a) and (7b) coincide with the solutions of the differential inclusions (8a) and (8b) respectively, we obtain the conclusion of the theorem ■

5 VPA for a class of discontinuous systems of fractional order

Let us next consider the following form of the IVP (1), where, for the sake of simplicity, the time variable t will be omitted hereafter

$$D_*^q x = f(x) := g(x) + \begin{pmatrix} \sum\limits_{\substack{i=1,\dots,n \\ j\in\{1,\dots,n\}}}^{n} \alpha_i^1 sign(x_i) x_j \\ \sum\limits_{\substack{i=1,\dots,n \\ j\in\{1,\dots,n\}}}^{n} \alpha_i^2 sign(x_i) x_j \\ \vdots \\ \sum\limits_{\substack{i=1,\dots,n \\ j\in\{1,\dots,n\}}}^{n} \alpha_i^n sign(x_i) x_j \end{pmatrix} \tag{10}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and α_i^j some real constants. Instead of *signum* function other piece-wise continuous functions, like Heaviside function, can be found in practical examples.
We will give the sufficient conditions so that VPA can be applied to systems modeled by the IVP (10).

Theorem 4 *Consider the IVP* (10). *Assume that* g *is homogenous and the following condition holds*

(*) *there exist* $\overline{i} \neq \overline{j}$, *such that* $\alpha_{\overline{i}}^{\overline{j}} \neq 0$.

Then, the unperturbed equations and equations perturbed via VPA, have the same solutions.

Proof The proof will be made at the time moments $j\Delta t$, $j = 1, 2, \dots$. Since g is homogenous $g[(1+\lambda)x] = (1+\lambda)g(x)$ and it is easy to see that $sgn[(1+\lambda)x_j] = sgn(x_j)$. Therefore, under the change of variable (3), Theorem 3 applies ■

Remark 2
(i) Condition (∗) in Theorem 4 ensures the discontinuity;
(ii) In [30] it is mentioned that the obtained stabilized orbits (or fixed points) are not identical to the underlying initially unstable periodic orbits (fixed points) which are embedded into a strange attractor, but was

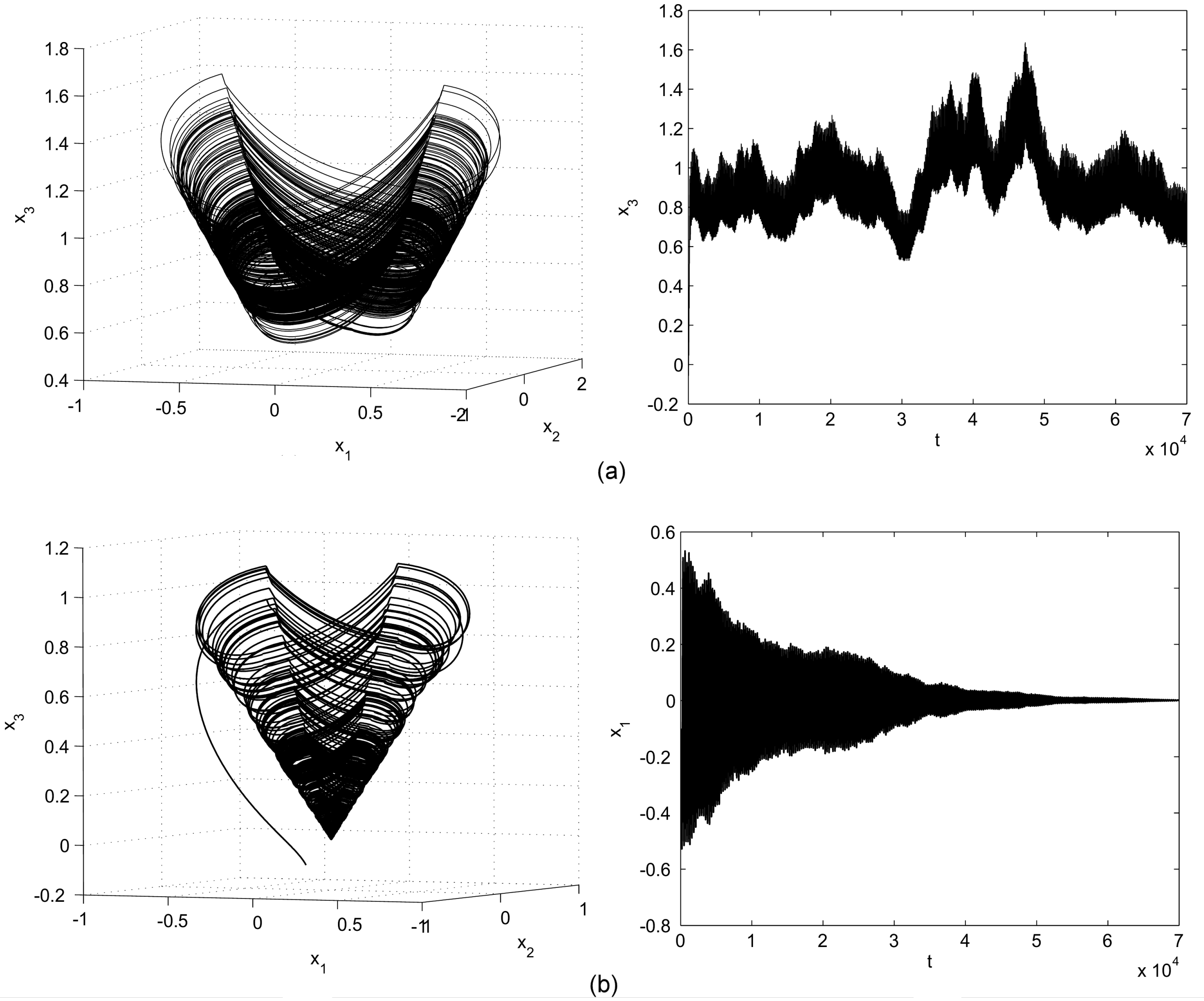

Fig. 3 (a) Chaotic attractor of the system (11). (b) Attractive stable fixed point of the system (11) obtained with VPA for $\Delta t = 0.02$ sec and $\lambda = -0.004$.

checked numerically that the stabilized orbits (fixed points) are close enough to the underlying ones of the unperturbed dynamical system for nearby parameter values corresponding to regular behavior.

Even less common, the applied systems modeled by (10) can be found e.g. in simple electronic circuits, such as the fractional variant of the piece-wise linear Chen system [43]

$$\begin{aligned} &D_*^q x_1 = a(x_2 - x_1), \\ &D_*^q x_2 = cdx_2 + (c - a - x_3)sgn(x_1), \\ &D_*^q x_3 = -bx_3 + x_1 sgn(x_2), \quad a, b, c \in \mathbb{R}^+, \end{aligned} \tag{11}$$

where, for $a = c$, we obtain a (10)-like system (there exist two pairs $(\bar{i}, \bar{j})$ verifying $(*)$: $(3, 1)$ in the $2nd$ equation and $(1, 2)$ in the $3th$ equation). For the order $q = 0.999$ and the parameter values $b = 0.265$, $c = a = 0.935$ and $d = 0.9$, the system behaves chaotically [1] (see the phase portrait and time series for x_3 in Figure 3 a).

To can apply VPA following the way exposed in above sections, the IVP (with $c = a$) is transformed into a set-valued one

$$\begin{aligned} &D_*^q x_1 = a(x_2 - x_1), \\ &D_*^q x_2 \in adx_2 - x_3 Sgn(x_1), \\ &D_*^q x_3 \in -bx_3 + x_1 Sgn(x_2). \end{aligned} \tag{12}$$

The discontinuity set is $M = \{(0, 0, x_3) | x_3 \in \mathbb{R}\}$, while the continuity set is $\mathbb{R}^3 \setminus M$. Next, we need to

[1] As known, systems of fractional order present chaotic evolutions not only in $\mathbb{R}^n$ for $n \geq 3$, but for q less than 1 too (see e.g. [44–46]), which means chaos in systems of fractional order persist for order lower than three (in our case $3 \times q = 2.997$).

integrate the IVP. For this purpose, in some chosen ε-neighborhood of discontinuity points, we shall determine a continuous single-valued selection, whose existence is ensured by Cellina's Theorem (see [38, Ch. 9, p. 358] or [39]). Due to the constructive characteristic of the proof, Cellina's Theorem allows the analytical construction of the selection. In this paper we used the way indicated in [47], where a smoothly cubic surface was chosen to approximate the $Signum(x_i)$ function in a neighborhood of x_i of ε radius (the simplest way is to choose the same ε radius for all discontinuity points). [2]

In this way, the set-valued IVP transformed into a fractional single-valued one which can be numerically integrated using e.g. the Adams-Bashoford-Moulton multi-step scheme for fractional differential equations proposed by K. Diethelm in [12].

While the IVP is numerically integrated, VPA requires to perturb the state variable every Δt with a small real quantity λ. For this example, we found that chosen $\Delta t = 2h$, where $h = 0.01$ sec is the integration step size, and $\lambda = -0.004$, the chaotic behavior is stabilized and the stabilized trajectory reaches the stable fixed point $(0, 0, 0)$ (Figures 3 b).

6 Discussion and open problems

In this paper we presented a theorem regarding the variables changes for a class of discontinuous systems of fractional order. The proof is done via a similar result for discontinuous systems of integer order presented in [27] via fractional differential inclusions. Next, we shown that this result explains why VPA can be applied to control the chaos in the class of considered discontinuous systems of fractional order modeled by (10).

We found that VPA also applies to systems of class (10) when f is no more homogenous such in the case $a \neq c$ or in the case of the nonhomogenous system (11) of integer order considered in [36]. Therefore, we could conclude that VPA requires very few restrictions and Theorem 4 gives only sufficient conditions to apply VPA.

An open problem is the study of the influence of the inherent history phenomenon induced by the multi-step characteristic of the numerical scheme (ABM) when the underlying IVP is integrated.

Another open question related to VPA applied to (10), regards the possible use of impulsive differential inclusions with fractional order, whose study dates very recently (see e.g. [48–50]), which could represent a more adequate approach.

[2] In numerical experiments, ε must be selected small enough, e.g. 10^{-3}, to be consistent with the physical characteristics of the system (see also [27, Ch. 2, p. 94]).

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