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Numerical Approximation of a Class of Discontinuous Systems of Fractional Order

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Abstract In this paper we investigate the possibility to formulate an implicit multistep numerical method for fractional differential equations, as a discrete dynamical system to model a class of discontinuous dynamical systems of fractional order. In this purpose, the problem is continuously transformed into a set-valued problem, to which the approximate selection theorem for a class of differential inclusions applies. Next, following the way presented in the book of Stewart and Humphries [1] for the case of continuous differential equations, we prove that a variant of Adams-Bashforth-Moulton method for fractional differential equations can be considered as defining a discrete dynamical system, approximating the underlying discontinuous fractional system. In this purpose the existence and uniqueness of solutions are investigated. One example is presented.

Keywords fractional systems · discontinuous systems · chaotic attractors · Filippov regularization · Adams-Bashforth-Moulton method for fractional differential equations

1 Introduction

Discontinuous differential equations modeling real phenomena, mainly in the field of dry friction mechanics, have been intensively studied especially after the occurrence of the pioneering work of Filippov [2] who had the idea to avoid the lack of classical solutions of this type os equations, by replacing the discontinuous single-valued initial value problem with a set-value one doted with enough regularity. Then, based on differential inclusions theory (see e.g. the known books of Aubin and Cellina [3] and Aubin and Frankowska [4]), a huge literature, motivated by mathematical and physical reasons, has been dedicated to discontinuous (Filippov) systems that arise mainly in mechanics (examples can be found in the book of Wiercigroch and de Kraker [5]) but also in other many field such as: chaotic circuits, convex optimizations, uncertain systems and so on.

On the other side, despite the fact that fractional derivatives date from 17th century (when to l'Hospital's famous question: "What does $\frac{d^n}{dx^n}f(x)$ mean if $n = 1/2$?" Leibniz answered: "It will lead to a paradox, from which one day useful consequences will be drawn"), they were not used than in the last few decades, when the use of fractional calculus allowed to study the enormous number of examples of systems in many domains such as: physics, chemistry, engineering, finance and so on (one of the early works being the book of Oldham and Spanier [6] or the papers of Caputo [7,8]). A possible explanation for this delay could be the fact that the fractional derivatives have no geometrical interpretation [9] or the fact that there are multiple definitions for fractional derivatives [10]. However, because fractional differential equations better describe the physical models, now the fractional derivatives has got focus (see references on their applications in [11]).

In this paper we are concerned with the "combination" of these categories of systems, namely to consider a class of *discontinuous systems of fractional order* which, on our knowledge, was not studied yet. Precisely, we are interested in the possibility to find a discrete dynamical system, generated by a numerical method for

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fractional differential equations, which models the underlying fractional discontinuous system.

Consider the discontinuous Initial Value problem (IVP) of fractional order

$$D_*^q x = f(x) := g(x) + As(x),$$
$$D_*^{q-k} x(0) = x_0^k,\ t \in I = [0,\infty),\ k = 0,1,\ldots,\lceil q\rceil - 1,$$

where we assume $g \in C(\mathbb{R}^n)$, $x(t) \in \mathbb{R}^n$ denotes a vector valued function of $t \in I$, $x_0^k \in \mathbb{R}$, A is $n \times n$ squared real matrix, and s is a piece-wise linear (discontinuous) function $s(x) = (sgn(x_1), sgn(x_2), ..., sgn(x_n))^T$. D_*^q, for q some positive real number, is the fractional operator chosen in this paper as being the Caputo operator with starting point 0 [6][1]

$$D_*^q u(t) = \frac{1}{\Gamma(q-n)} \int_0^t \frac{u^{(n)}(\tau)}{(t-\tau)^{q-n+1}} d\tau,$$

for $n-1 < q < n$, and $\Gamma(m)$ the factorial (Gamma) function given by the following expression

$$\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx,$$

for which, when m is an integer, it holds that

$$\Gamma(m+1) = m!$$

$D_*^{\lceil q\rceil}$ stands for the standard differential operator. Because in most applications in science and engineering $q \in (0,1]$, we have to specify in this case in the IVP just one initial condition. For practical reasons, it is usually to specify the initial conditions under the integer derivatives form (way proposed by Caputo in [7] fact which represents another reason to use Caputo operator). These data typically have a well understood physical meaning and can be measured (more details on the choice of initial conditions may be found in [12] and [13]). Thus. the IVP becomes

$$D_*^q x = f(x) := g(x) + As(x), \tag{1}$$
$$x^{(k)}(0) = x_0^{(k)},\ t \in I,\ k = 0,1,\ldots,\lceil q\rceil - 1,$$

Remark 1 i) Since $f(x)$ does not depends explicitly on t, we can use use homogeneous initial conditions (given at $t = 0$);
ii) The IVP (1) represents a general case since, function of A and q, it embeds three common cases:

-If all entries in A are zero and q is integer, the IVP models a continuous dynamical system;

-If all entries in A are zero and q is a fractional number, the IVP models continuous fractional dynamical systems;

-In the rest of the cases, the IVP defines dynamical systems discontinuous with respect to the state variable.

[1] There are many ways to define a fractional differential operator (see e.g. [8]). We have chosen here the Caputo operator as being the most used in practical problems.

2 Continuous approximation of IVP (1)

The existence and uniqueness of solutions to discontinuous IVPs (of fractional or integer order), such as (1), are essential because, due to the right-hand discontinuity, the classical solutions of IVP might not even exist. For discontinuous vector fields, the existence and uniqueness of solutions is not guaranteed in general, no matter what notion of solution is chosen. Moreover, the classical notion of solution for ODEs is too restrictive in this case. A possible solution to encompass this difficulty is to shift the single valued IVP into a set-valued one, namely a differential inclusion, solution given by Filippov in [2] using a generalized concept of solution

$$D_*^q x \in F(x),$$

where $F : \mathbb{R}^n \Longrightarrow \mathbb{R}^n$ is a set-valued vector function defined on the set of all subsets of $\mathbb{R}^n$. The simplest definition of F is the following convex form [2]

$$F(x) = \bigcap_{\varepsilon>0} \bigcap_{\mu(M)=0} \overline{conv} f\left((x+\varepsilon B)\setminus M\right), \tag{2}$$

where, M is the set of discontinuity points of f, B the unit ball in $\mathbb{R}^n$, μ the Lebesgue measure. At the points where the function f is continuous, $F(x)$ will consist of one point, which is the value of f at this point, while at the discontinuity points, $F(x)$ is the convex hull of values of $f(x^*)$, $x^* \in M$, given by (2) ignoring the behavior on null sets.
For the case $n = 1$, the Filippov regularization applied to the sign function leads to the set-valued (sigmoid) function

$$Sgn(x) = \begin{cases} \{-1\} & \text{for } x < 0 \\ [-1,1] & \text{for } x = 0, \\ \{+1\} & \text{for } x > 0 \end{cases}$$

Applying the Filippov regularization to the right hand side of the IVP (1) one obtains the following setvued IVP of fractional order

$$D_*^q x \in F(x) := g(x) + ASgn(x), \tag{3}$$
$$x^{(k)}(0) = x_0^{(k)},\ k = 0,1,\ldots,\lceil q\rceil - 1,\ \text{for a.a. } t \in I.$$

Remark 2 The next notions and results are presented for the metric space $\mathbb{R}^n$ since the most applications are defined in this space, but they are true in general metric spaces.

Definition 1 A *selection* of a given set-valued function $F : \mathbb{R}^n \Longrightarrow \mathbb{R}^n$ is a function $h : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$h(x) \in F(x), \quad \forall x \in \mathbb{R}^n.$$

In [15] the following property for the right-hand side of (3) is proved

Proposition 1 *The set-valued function F in (3) is upper semicontinuous with closed and convex values.*

The next result is known as Approximate Selection Theorem or Cellina's Theorem [3,4]

Theorem 1 *Let $F : \mathbb{R}^n \Longrightarrow \mathbb{R}^n$ be an upper semicontinuous. If the values of F are nonempty and convex, then for every $\varepsilon > 0$, there exists a locally Lipschitz single valued function $h_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$Graph(h_\varepsilon) \subset B(Graph(F), \varepsilon).$$

Remark 3 Cellina's Theorem asserts too that the graph of h_ε is embedded in an "$\varepsilon - tube$" around the graph of F (i.e. belongs to the convex hull of the image of F). Moreover, because the proof is constructive, it allows to determine h_ε for the practical examples (selection strategies can be found in [14]).

Theorem 2 *The set-valued IVP (3) with g continuous admits a locally Lipschitz selection.*

Proof It is easy to verify that F is upper semicontinuous and has closed and convex values due to the symmetric interpretation of a set-valued function as a graph [15]. Thus, F verifies the conditions in the Cellina's Theorem 1 and therefore, admits a locally Lipschitz selection.

Thus, the set-valued initial value problem (3) transforms into the following single-valued continuous IVP of fractional order

$$D_*^q x = h_\varepsilon(x),\ x^{(k)}(0) = x_0^{(k)},\ k = 0, 1, \ldots, \lceil q \rceil - 1,\ t \in I. \tag{4}$$

3 Numerical solutions of fractional differential equations

Consider next, for the sake of simplicity, the scalar form of IVP (4). However, all the results can be extended to systems of equations without any problems.

Theorem 3 *The IVP (4) admits a unique solution on I.*

Proof h_ε being locally Lipschitz, the classical theorem for existence and uniqueness applies. (see also [11, Chapter 6] where the existence and uniqueness for general fractional equations are treated)

There are only a few numerical methods for fractional differential equations (see e.g. [10,16]). Also, there are some frequency domain techniques based on Bode diagrams, which allow to obtain a linear approximation for the fractional-order integrator [17]. However, because it is not clear if they can be generalized, we focus in this paper on a variant of the classical Adams–Bashforth–Moulton (ABM) integrator that has been constructed and analyzed in [12] for fully general sets of equations without any special assumptions. Moreover, the method is easy to implement computationally.

We assume that we are working on the time interval $[0, T]$, $T > 0$, partitioned by the equispaced grid: $\{t_0, t_1, \ldots, t_N\}$ with $t_n = nh$, N some positive integer and $h = T/N$. Denote by x_j the numerical approximation of $x(t_j)$ for $j = 0, 1, \ldots, n$.

The predictor phase (the fractional Adams–Bashforth method) it then first computes a preliminary approximation x_{i+1}^p

$$x_{i+1}^p = \sum_{j=0}^{\lceil q \rceil - 1} \frac{t_{i+1}^j}{j!} x_0^{(j)} + \frac{1}{\Gamma(q)} \sum_{j=0}^{i} b_{j,i+1} f(x_j), \tag{5}$$

where

$$b_{j,i+1} = \frac{h^q}{q} \left[(i + 1 - j)^q - (i - j)^q \right].$$

Then, the corrector phase (the fractional variant of the one-step Adams–Moulton method) determines the actual final approximation x_{i+1} which is

$$x_{i+1} = \sum_{j=0}^{\lceil q \rceil - 1} \frac{t_{i+1}^j}{j!} x_0^{(j)} + \tag{6}$$

$$\frac{h^q}{\Gamma(q+2)} \left(\sum_{j=0}^{i} a_{j,i+1} f(x_j) + f\left(x_{n+1}^p\right) \right),$$

with

$$a_{j,n+1} = \begin{cases} n^{q+1} - (n - q)(n + 1)^q & \text{if } j = 0, \\ (n - j + 2)^{q+1} + (n - j)^{q+1} & \\ -2(n - j + 1)^{q+1} & \text{if } 1 \leq j \leq n, \\ 1 & \text{if } j = n + 1. \end{cases}$$

Thus, the fractional ABM method is given by the equations (5) and (6) being a variant of the classical second-order ABM method.

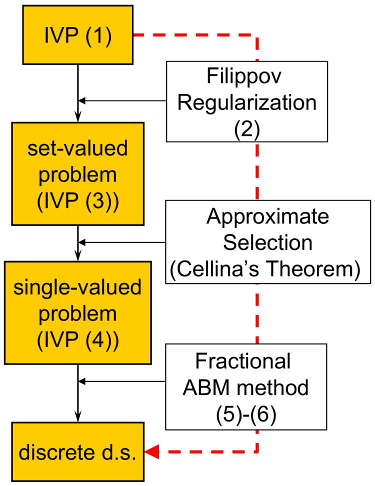

Fig. 1 The steps to obtain the discrete numerical approximation.

There are several ways to approximate Gamma function Γ, the most utilized being the Lanczos approximation [18].

The stability properties are at least as good as the corresponding properties of the classical second-order ABM method assuring the convergence of the invariant sets of the fractional ABM method to the real invariant sets of underlying system.

Remark 4 In contrast to differential operators of integer order, fractional derivatives are not local operators. Therefore, to approximate $D_*^q u(t)$ we have to take the entire history of u (i.e., all function values $u(\tau)$ for $0 \leq \tau \leq t$) into account. This is an impediment due to significantly higher computational effort. However, this property is highly desirable from the physical point of view because it allows us to model phenomena with memory effects.

4 Numerical approximation of the IVP (1)

In this section, following the idea presented in [1] for continuous systems, where it is proved that an implicit multistep numerical method posses invariants sets (like attractive fixed points, limit cycles or chaotic attractors) which, via convergence property, may approach the real invariant sets, we prove that this way can be adopted for our class of systems.

In this purpose consider the ABM scheme (5)-(6) in the following form

$$H\left(x_{n+1}, x_n\right) = 0, \quad x_0 = x(0), \tag{7}$$

where $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$.

Definition 2 [1] Equation (7) is said to define a dynamical system on a subset $E \subseteq \mathbb{R}^n$ if, for every $x_0 \in E$, there exists a unique solution $y \in E$ of $H(y, x_0) = 0$.

Remark 5 [1] When the solution of (7) is not unique, Definition 2 gives the notion of so called *generalized dynamical systems.*

The next theorem is the main result of this paper which proves that the ABM method for fractional differential equations may be viewed as a discrete dynamical system which models our class of systems

Theorem 4 *The ABM scheme (5)-(6) applyed to the IVP (4) defines a dynamical system.*

Proof The existence and uniqueness of solution to (4) is given by Theorem 3. Also, due to the convergency of the fractional ABM method (5)-(6) (see for convergency e.g. [13]), equation (7) admits a unique solution. Therefore, by Definition 2, the fractional ABM method (5)-(6) defines a dynamical system.

Remark 6 We can consider that Theorem 4 indicates how a fractional system can be numerically approximated.

Summarizing, the steps which lead to numerical approximation of IVP (1) and defines a discrete dynamical system, are depicted in Fig.1.

Example

Let us consider the fractional discontinuous variant of the Chen's system [19][2]

$$\begin{cases} D_*^q x_1 = a\left(x_2 - x_1\right) - 0.5sgn(x_1), \\ D_*^q x_2 = x_1(c-a) - x_1 x_3 + c x_2, \\ D_*^q x_3 = x_1 x_2 - b x_3, \end{cases} \tag{8}$$

with $a = 35, \quad b = 3$ and $c = 28$.

[2] which can be considered as being a *fractional jerk system* (see [20]).

Here

$$g(x) = \begin{pmatrix} a(x_2 - x_1) \\ x_1(c-a) - x_1x_3 + cx_2 \\ x_1x_2 - bx_3 \end{pmatrix},$$

and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Two chaotic attractors (for $q = 1$ and $q = 0.8$) are plotted in Fig.2 a and b, while in Fig.2 c a stable fixed point is presented. The utilized step size is $h = 0.005$, the number of time steps $N = 10,000$ and the initial data were the same for each case. To approximate the first component of the right-hand side of (8), denoted by $f_1(x_1, x_2)$, a cubic surface was utilized following the algorithm proposed in [21]. The graph of the set-valued function F_1, before and after his approximation, is presented in Fig.3a,b.

Remark 7 As known yet, chaos may be found at (fractional) systems of order less than 3 and not as believed before when, due to the Poincaré-Bendixon theorem, the nonlinear systems of integer order must have a minimum order of 3 to display chaotic motion. For our example chaos persists while q is set about $q = 0.793$, when the order of system is $3 \times 0.793 = 2.379$ and the system stabilizes the trajectory which is attracted to one of his two fixed points (Fig.2 c). For the continuous fractional Chua's system, chaotic attractors appears for order as low as 2.7 [22]. Moreover, chaos may appears even in systems with order less than 2, as in the case of nonautonomous Duffing systems of fractional order [23].

5 Comments

In this paper is discussed the possibility to consider the fractional ABM method as a discrete dynamical system for a class of fractional discontinuous systems modeled by the IVP (1), after which the IVP was first approximated using Cellina's Theorem.

Another possibility to approach these systems is to deal with the fractional differential inclusions obtained by the Filippov regularization (see e.g. the recent paper of Chang and Nieto [24] for the existence of solutions of fractional differential inclusions with boundary value conditions, or [25]). However, for the practical examples, the numerical approach proposed here is more convenient.

A still open problem occurring when a numerical method for some class of continuous or discontinuous (of fractional or integer order) system is to realize a qualitative and comparative study of the dynamics of both systems, the underlying and the approximated one.

Another open problem is to investigate the possible influence of the discontinuity on the system order reduction when chaos disappears.

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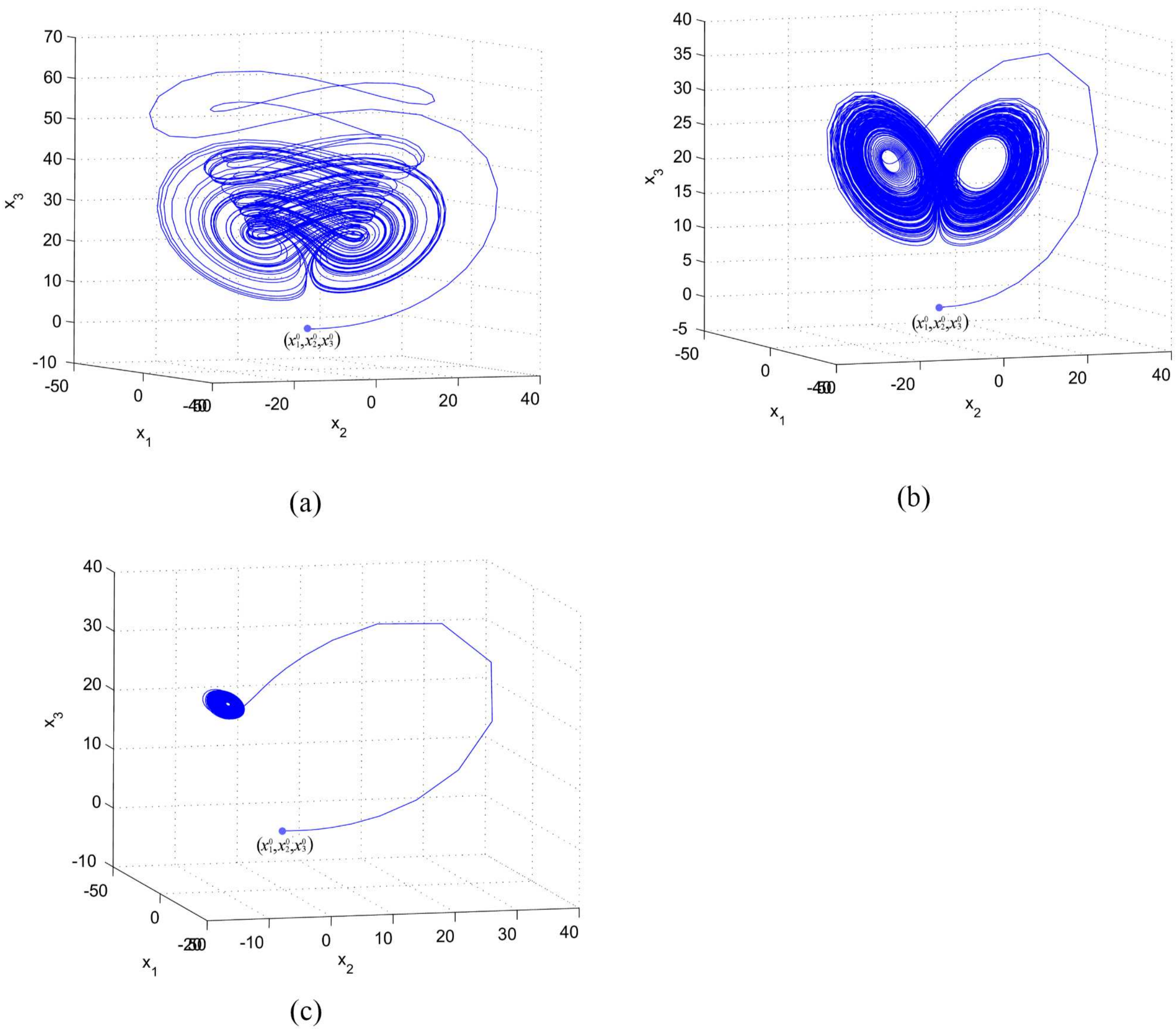

Fig. 2 Three attractors of Chen's system (8). a) Chaotic attractor for $q = 1$; b) Chaotic attractor for $q = 0.8$; c) Stable attractor for $q = 0.793$.

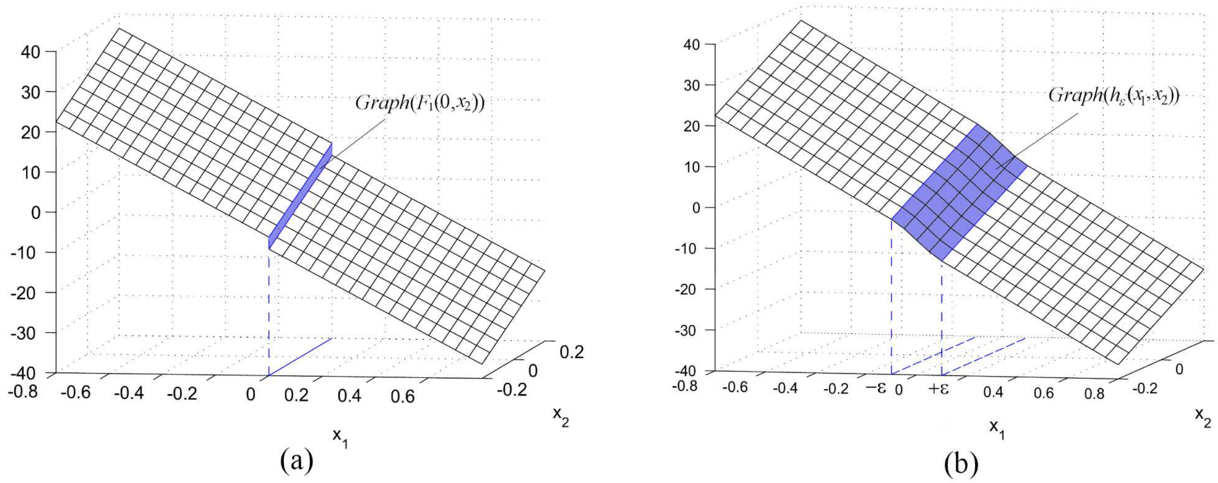

Fig. 3 The graph of $f_1(x_1, x_2)$. a) Before approximation; b) After approximation.

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