ON NUMERICAL INTEGRATION OF DISCONTINUOUS DYNAMICAL SYSTEMS

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This paper addresses an important issue in numerical integration of dynamical systems, integer or fractional order, with discontinuous vector fields. It is shown that these systems cannot be solved using numerical methods designed for ODEs with continuous functions on the right-hand side, therefore have to resort to special schemes and procedures in numerical integrations such as continuous approximations of the right-hand sides of the ODEs.

Keywords: Chaos, Lyapunov exponent, fractional order, initial value problem, discontinuous dynamical system

1. Introduction

As is well known, to solve most nonlinear ODEs it is only possible by numerical integration, especially for those having discontinuous functions on the right-hand sides of the equations with respect to the state variables.

For a dynamical system with continuous right-hand side, simply called a *continuous system* in this paper, it is usually expressed as the following continuous Initial Value Problem (IVP):

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1}$$

where $f : D \subset \mathbb{R} \times \mathbb{R}$ is a continuous function, $x_0 \in \mathbb{R}$, for which the Péano theorem ensures the existence of a solution.

Theorem 1. *If f is continuous on D, then the IVP* (1)*, with $(t_0, x_0) \in D$, admits at least one local solution $x : I \to \mathbb{R}$, which solves* (1) *for all $t \in I$, where I is a neighborhood of t_0.*

Now, consider for simplicity the autonomous[1] dynamical systems modeled by the following differential equations with a discontinuous function on the right-hand side (simply called a *discontinuous system* in this paper):

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \in I = [0, T], \tag{2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a discontinuous function and $x_0 \in \mathbb{R}^n$ and $T > 0$.

Recall that the continuity is important, which creates a dichotomy between the two classes of continuous systems and discontinuous systems. Thus, beside the fact that a discontinuous system may exhibit phenomena like sliding motion, corner-collision and grazing-sliding bifurcations (which appears mostly in mechanical systems with dry friction, see e.g. [Wiercigroch & de Kraker, 2000]), there are few more differences between these two classes of systems: the seemingly most important difference is that, while most of IVPs admit at least one solution and therefore can be directly numerically integrated by using a standard numerical integrator implemented by e.g. Matlab or Mathematica, the situation changes drastically in the case of discontinuous systems, which require special numerical methods since the common numerical methods designed for continuous systems do not work in general. In fact, it is difficult to define a solution when the orbit passes through a discontinuity surface.

In this paper, we are interested in the existence of solutions to (2)*, and its correct numerical solution. For this purpose, Matlab integrators will be utilized.*

Consider, for example, the following discontinuous system [Filippov, 1988]:

$$\dot{x} = 1 - 2\,\mathrm{sgn}(x) = \begin{cases} 3 & x < 0, \\ 1 & x = 0, \\ -1 & x > 0, \end{cases} \tag{3}$$

This system has no classical solutions. Thus, if $x_0 \neq 0$, a solution is (Fig. 1 (a))

$$x(t) = \begin{cases} 3t + x_0, & x < 0, \\ -t + x_0, & x > 0. \end{cases}$$

As t increases, each of these solutions tends to the line $x = 0$, because the vector field determined by $\dot{x}$ in both half-planes forces the solution orbit to remain on this line but never leave upwards or downwards. On the other hand, once a solution has arrived on the line $x = 0$ (at time moments of $t' = -x_0/3$, for $x_0 < 0$, and $t'' = x_0$ for $x_0 > 0$) cannot continue to move along this line, because the solution so obtained is $x(t) = 0$, which does not satisfy the equation in the usual sense: its derivative is $\dot{x}(t) = 0$, while the function on the right-hand side gives $1 - 2\,\mathrm{sgn}(0) = 1$. On the other hand, a solution to this equation in the "classical sense" should verify the IVP everywhere on its own interval of existence, which is not satisfied by this example.

Remark 1.1. Any numerical method for solving continuous ODEs will face with difficulties when the equation presents discontinuities on the right-hand side and stiff phenomenon may arise. Consider, for example the Matlab integrator ode45. Jan Simon [Simon, 2017] pointed out that using ode45 integrator (or other routines for continuous systems) sometimes can reach a final value, but from the view point of a scientist working in the field of numerical computations, this final value cannot be considered as a real "result". For such systems, the numerical integration in the neighborhood of the discontinuity, using numerical methods for continuous systems, is a kind of measurement process based on an extremely large number of smaller and smaller steps (see Example 6 below), which cause round-off errors and local discretization errors. Also, the step-size control of the routine ode45 can lead to unexpected effects and the solver might integrate right over a discontinuity without noticing this. In these cases, the results have poor accuracies, which are highly doubtful especially in the neighborhoods of discontinuities. Matlab ode45 will reduce the step size to such a tiny value that the integration could take hours and hours to run while the accumulated rounding errors dominate the solutions. Similarly, in Mathematica, NDSolve cannot deal

[1] the results can be extended to nonautonomous systems.

with discontinuities without special treatments; otherwise, all Mathematica's solvers degrade by reducing drastically the accuracies.

2. A possible way to integrate the discontinuous IVP (2)

The numerical solutions of IVPs modeling differential equations with discontinuous functions on the right-hand sides can be traced back to the work of Taubert [Taubert, 1973, 1981], which was further investigated by several authors such as Kastner-Maresch [Kastner-Maresch, 1992], Lempio [Lempio & Veliov, 1998] (see also the review of Dontchev and Lempio [Dontchev & Lempio, 1992] or [Stewart, 1990; Elliot, 1985; Kastner-Maresch, 1990-91; Cortés, 2008; Lempio, 1993; Dieci & Lopez, 2012; Kunze & Küper, 1997; Kunze, 2000]). For discontinuous systems in some particular forms, Stewart [Stewart, 1990] transformed the discontinuous right-hand side on the manifold of discontinuity to a classical Lipschitz continuous function.

"Event detection" (numerically interception of the discontinuity manifold) is another approach being utilized in some numerical methods for solving discontinuous systems. For example, a coarse solution would be to identify the point in time when a discontinuity occurs and abort the integration at exactly this point. Next, the integration will be restarted from this point on, but ctually on a different differential equation due to the change of the right-hand function.

There are also dedicated software such as *Siconos: non-smooth numerical simulation*, an open-source scientific software primarily targeting modeling and simulating nonsmooth dynamical systems using C++ or Python [Siconos, 2017].

The numerical routines used in this section are Matlab $ode45$ while the routine designed for stiff systems is $ode23s$.

One simple way to deal with discontinuous systems, which can be easily implemented computationally, is to approximate the discontinuous function by a continuous function.

Theorem 2. *[Danca et al., 2017] The discontinuous IVP* (2) *can be approximated by the continuous IVP*

$$\dot{x}(t) = \tilde{f}(x(t)), \quad x(0) = x_0, \tag{4}$$

where $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function.

Proof. *(Sketch) The underlying discontinuous IVP is transformed to a set-valued problem, via Filippov regularization [Filippov, 1988]. Next, Cellina's Theorem [Aubin & Cellina, 1984, p. 84] shows that the set-valued problem admits a Lipschitz continuous approximation $\tilde{f}$.* ■

For the discontinuous sgn function, there are several smooth (sigmoid) approximations, $\tilde{f}$, such as[2]

$$\widetilde{\text{sgn}}(x) = \frac{2}{1 + e^{-\frac{x}{\delta}}} - 1,$$

where δ is a positive parameter controlling the slope of the curve in the neighborhood of the discontinuity $x = 0$ (see Fig. 2, where $\widetilde{\text{sgn}}$ is plotted for three distinct values of δ). In applications, $\delta = 1e - 6$ proves to be an adequate choice (see [Danca et al., 2017] for a detailed study regarding the values of δ).

Note that since Lipschitz functions are smooth, Theorem 2 ensures the existence of smooth approximations, such as sigmoid-type functions. For example, Theorem 2 enables the transformation of the discontinuous system (3) to the following continuous (even smooth) one:

$$\dot{x} = 1 - 2\widetilde{\text{sgn}}(x) = 3 - \frac{4}{e^{-\frac{x}{\delta}}}, \tag{5}$$

which admits a unique solution on $t \in [0, \infty)$. Thus, the solution can be extended along the line $x = 0$ (Fig. 1 (b)).

Once the underlying IVP is transformed to a continuous IVP, it can be numerically integrated using any numerical method for continuous ODEs.

[2] see [Danca et al., 2017] for other possible approximations.

Consider the following example of a theoretical stiff non-autonomous discontinuous model [Kastner-Maresch, 1992, 1990-91]:

$$\begin{aligned}\dot{x}(t) &= 5(h(t)-x(t))+h'(t)\\ &+5-5\,\mathrm{sgn}(x(t)),\quad t\in[0,\infty),\end{aligned}\tag{6}$$

with

$$h(t)=-\frac{4}{\pi}\,\mathrm{atan}(t-1).$$

In [Kastner-Maresch, 1992], it is proved that for $t\in[0,1]$ the exact solution from $x(0)=1$ is $x(t)=h(t)$. For $t>1$, because the vector field is negative for $x>0$ and positive for $x<0$, the solution will stick in the discontinuity manifold $x=0$ (similar to the example (3)). Therefore, the exact classical solutions on $t\in[0,1]$ and $t>1$ are

$$x(t)=\begin{cases} h(t),\ t\in[0,1],\\ \ \ 0,\quad t>1.\end{cases}$$

After applying the continuous approximation, by replacing sgn with $\widetilde{\mathrm{sgn}}$ (Theorem 2), one can find the numerical approximation of the solution on $[0,\infty)$, by integrating the IVP:

$$\begin{aligned}\dot{x}(t) &= 5(h(t)-x(t))+h'(t)\\ &+5-5\widetilde{\mathrm{sgn}}(x(t)),\quad t\in[0,\infty),\end{aligned}\tag{7}$$

using a standard numerical scheme for continuous ODEs.

Consider $t\in[0,2]$. In Fig. 3 (a), for initial condition $x(0)=1$, the exact solution, the numerical solution of the discontinuous system (6) and the numerical solution of the continuous system (7) with $\delta=1e-6$, are overplotted, which are obtained using the Matlab $ode45$ integrator. Matlab $ode45$ is used for both systems in order to underline the incorrect utilization of integrators like $ode45$ for discontinuous systems. Even apparently both solutions of the discontinuous system (star plot) and the continuous approximated system (circle plot) seem to be identical to the exact solution (dotted plot), there exist some differences (see the zoomed detail in the top-right parts of Fig. 3 (a) and Fig. 3 (b)). Computational times with a dual core processor and with the 64 bits Matlab 8, are about $54.24s$ for system (6) and $66.83s$ for (7), respectively, while the calculated points are 378973 and 508158, respectively. This difference is due to the supplementary calculations for $\widetilde{\mathrm{sgn}}$ compared to sgn.

Since discontinuity generally induces sliding phenomena, it is useful to examine the performances, such as the computing time, if one uses dedicated routines for stiff ODEs. Thus, using Matlab $ode23s$ integrator to calculate the solution to the continuous system (7) needs about $0.028s$, while the number of determined points for (6) is only 40.

In the sliding region, $t\in[1,2]$, the mean value of the solution of the approximate system (6) is $2.9846e-07$, while for the non-approximated system (7) it is $2.1411e-06$, reflecting a better solution of the approximated system compared with the exact solution $x(t)=0$. However, because $ode23s$ determines only about 8 points of the solution in the sliding region, the sliding dynamics are not accurately represented and the mean value of the solution is larger: $9.1033e-05$. Also, $ode23$ produces solution points far from the exact solution even in a non-sliding region, $t\in[0,1]$ (see the green line in the upper-right part of Fig. 3 (a)). A similar conclusion can be drawn with the other commonly used Matlab routine, $ode15s$.

The difference between “correct numerical solutions” of system (4) and “wrong numerical solutions” of system (2) can lead to different and even unexpected dynamical behavior of nonlinear discontinuous dynamical systems. For example, consider the following variant of Chua's circuit [Brown, 1993]:

$$\begin{aligned}\dot{x}_1 &= -2.57x_1+9x_2+3.86\,\mathrm{sgn}(x_1),\\ \dot{x}_2 &= x_1-x_2+x_3,\\ \dot{x}_3 &= -px_2,\end{aligned}\tag{8}$$

where p is the bifurcation parameter. In Fig. 4 (a), the component values x_1 of the wrong solution to the discontinuous (red plot) and of the correct solution of the approximated (blue plot) systems are obtained with *ode*45 for the same initial conditions and $p = 15$. It can be seen that, after a relatively short time, about $t \approx 25$ in the graph, and even after a shorter time in the numerical computation, the two solutions become different.

3. Integrating discontinuous systems of fractional order

To the best of our knowledge, there exist no theoretical frameworks for systems modeled by differential equations of fractional order (FDEs) with discontinuous functions on the right-hand side, and likewise there exist no numerical methods for solving them. Although they have not been rigourously proved and analyzed, systems of fractional order (FO), modeled by discontinuous functions, could have better physical meanings for some real systems. Since the solution of a discontinuous system of integer order represents a vector field crossing some switching surfaces can be locally analyzed, the whole dynamics can be composed by locally-defined flows. For discontinuous systems of FO, however, this is not possible. Actually, for discontinuous systems of FO, transversally crossing or sliding solutions have not yet been rigorously analyzed. Similarly, the existing theory of Lyapunov exponents for classical dynamical systems remains to be generalized to discontinuous systems of integer order, and then to that of FO. Here, once again, using routines designed for continuous systems to calculate Lyapunov exponents of discontinuous system, integer order or FO, could be totally wrong.

Specifically, FDEs do not define dynamical systems in the usual way. Consider the following general discontinuous IVP of FO:

$$D_*^q x(t) = f(x(t)), \quad x(0) = x_0, \quad t \in I = [0, \infty), \tag{9}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is some discontinuous function and D_*^q is the Caputo differential operator of FO q, with starting point 0 [Podlubny, 1999]:

$$D_*^q x(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-\tau)^{-q} x'(t) d\tau \,.$$

By denoting the solution of (9) as $\Phi(t, x_0)$, one does not have the classical flow property $\Phi_s \circ \Phi_t = \Phi_{t+s}$ [Zhou, 2016]. However, by numerical calculation of the underlying solutions, the definition of an integer-order dynamical system is adopted, which states that if the underlying IVP admits a solution then the problem defines a dynamical system ([Stuart & Humphries, 1998, Definition 2.1.2]).

Although there exist numerical methods for FDEs (see e.g. [Diethlem et. al, 2002; Diethlem, 2003]), to the best of our knowledge, there are no numerical methods for FDEs with discontinuous functions on the right-hand sides.

Therefore, for FDEs with discontinuous right-hand sides, obtaining numerical approximations to their solutions is a difficult task. Here, using standard routines designed for integer-order FDEs will certainly lead to incorrect results.

Nevertheless, the trouble can be handled. A possible approach is, as described in Section 2, using continuous (even smooth) approximations. Specifically, the discontinuous IVP of FO is transformed to a continuous one, then some routines such as the predictor-corrector multi-step Adams-Bashforth-Moulton method [Diethlem, 2003] can be utilized.

Remark 3.1. Similarly to discontinuous systems of integer order, numerically finding local Lyapunov exponents of discontinuous systems of FO requires special care because, beside the necessary (approximated) continuity of the function on the right-hand side, the smoothness of the corresponding Jacobian matrix must also be ensured.

Consider the fractional-order variant of the system (8), of order $q = 0.99$, as follows:

$$\begin{aligned} D_*^{0.99} x_1 &= -2.57x_1 + 9x_2 + 3.86\,\mathrm{sgn}(x_1), \\ D_*^{0.99} x_2 &= x_1 - x_2 + x_3, \\ D_*^{0.99} x_3 &= -px_2. \end{aligned} \tag{10}$$

Using the same initial conditions and $p = 15$, the components x_1 obtained with the Adams-Bashforth-Moulton method for the discontinuous system and the approximated system are overplotted in Fig. 4

(b) for visualization. Here, the difference between the two solutions appears later than in the case of the integer-order system.

4. Conclusion

In this paper, we have underlined the common and potential mistakes when numerically solving dynamical systems modeled by ODEs with discontinuous right-hand side, of integer or fractional order, by using methods and routines designed for continuous ODEs. A possible way to deal with this problem is to use a continuous approximation to the discontinuous function on the right-hand side of the system, with a sigmoid-like function. After the approximation, the IVP can be numerical integrated with common numerical methods designed for continuous differential equations.

Compared with other methods, which require codes related to generally complicated algorithms (such as discontinuity interception, which runs different codes in continuous and discontinuous regions respectively), the continuous approximation approach presented in this paper requires only the simple replacement of the discontinuous function with a sigmoid-like function.

References

Aubin, J.- P. & Cellina, A. [1984] *Differential Inclusions: Set-Valued Maps and Viability Theory* (Springer, Berlin).

Brawn, R. [1993] "Generalizations of the Chua Equations," *IEEE Trans. on Circ. Sys.* **40** (11), 878-883.

Cortés, J. [2008] "Discontinuous dynamical systems: A tutorial on solutions, nonsmooth analysis, and stability," *IEEE Contr. Syst. Mag.* **28**, 3673.

Danca, M-F., Feçkan, M., Kuznetsov, N. V., Chen, G. [2017] "Complex dynamics, hidden attractors and continuous approximation of a fractional-order hyperchaotic PWC system," submitted.

Dieci, l, & Lopez, L. [2012] "A survey of numerical methods for IVPs of ODEs with discontinuous right-hand side," *Journal of Computational and Applied Mathematics* **236**, 39673991.

Diethelm, K. [2003] "Efficient solution of multi-term fractional differential equations using P(EC)^{m}E methods," *Computing*, **71**(4), 305–319.

Diethelm, K., Ford, N.J. & Freed, A.D. [2002] "A predictor-corrector approach for the numerical solution of fractional differential equations," *Nonlinear Dynamics*, **29**(1),3–22.

Dontchev, A. & Lempio, F. [1992] "Difference methods for differential inclusions: A survey," *SIAM Rev.* **34**, 263294.

Elliot, C. M. [1985] "On the convergence of a one-step method for the numerical solution of an ordinary differential inclusion," *IMA Journal of Numerical Analysis* **5**, 3-21.

Filippov, A. F. [1988] *Differential Equations with Discontinuous Righthand Sides* (Springer, Berlin).

Kastner-Maresch, A. E. [1992] "The implicit midpoint rule applied to discontinuous differential equations," *Computing* **49**, 4562.

Kastner-Maresch, A. E. [1990-91] "Implicit Runge-Kutta methods for differential inclusions," *Numer. Funct. Anal. and Optimiz.* **11**, 937-958.

Kunze, M. & Küper, T. [1997] "Qualitative bifurcation analysis of a non-smooth friction-oscillator model", *Zeitschrift für angewandte Mathematik und Physik ZAMP* **48**, 87-101.

Kunze, M. [2000] *Rigorous Methods and Numerical Results for Dry Friction Problems, in Applied Nonlinear Dynamics and Chaos of Mechanical Systems with Discontinuities,* Wiercigroch, M & de Kraker, B. eds. (World Scientific, Singapore).

Lempio, F. [1993] "Modified Euler methods for differential inclusions," in: A. B. Kurzanski, V. M. Veliov, Set-valued Analysis and Differential Inclusions. A Collection of Papers resulting from a Workshop held in Pamporovo, Blugaria, September 1990 (Birkhäuser, Boston-Basel-Berlin).

Lempio, F. & Veliov, V. [1998] "Discrete approximations of differential inclusions," *Bayreuth. Math. Schr.* **54**, 149232.

Podlubny, I. [1999] *Fractional Differential Equations* (Academic Press, San Diego).

O'Regan, G. P. [1970] "Step size adjustment at discontinuities for fourth order Runge-Kutta methods," *Computer Journal* **13** 10.1093/comjnl/13.4.401.

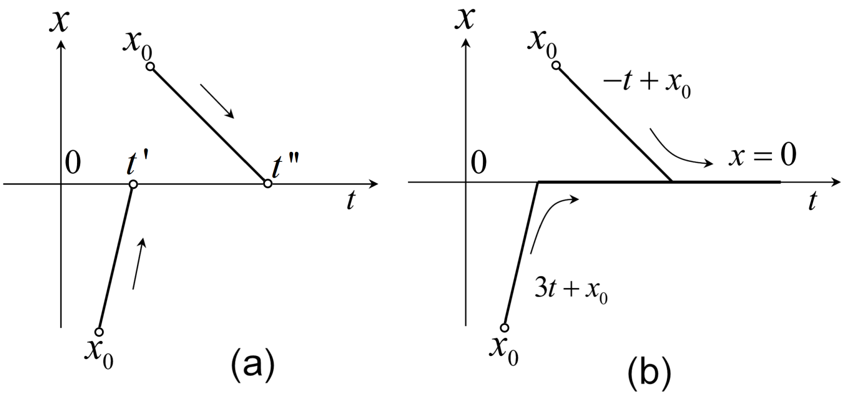

Fig. 1. (a) Graph of solution of (3). (b) Graph of generalized solution of the approximated problem (5).

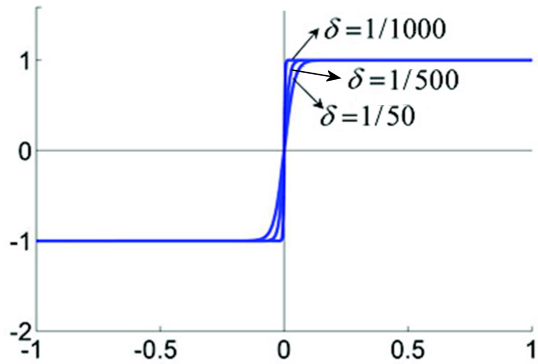

Fig. 2. Graph of the sigmoid function (2), for three values of δ: 1/50, 1/500 and 1/1000. Smaller δ values give better approximations of the discontinuous sgn function.

Stewart, D. [1990] "A high accuracy method for solving ODEs with discontinuous right-hand side," *Numer. Math.* **58**, 299-328.

Simon, J. [2017] *Matlab Answers:* https://www.mathworks.com/matlabcentral/profile/authors/869888-jan-simon; https://www.mathworks.com/matlabcentral/answers/127971-ode-45-integration-control

[2017] *Siconos: Nonsmooth numerical simulation,* http://siconos.gforge.inria.fr/4.1.0/html/index.html

Stuart, A. & Humphries, A. R. [1998] *Dynamical Systems and Numerical Analysis* (Cambridge University Press).

Taubert, K. [1973] "Uber die Approximation nicht-klassischer Lösungen von Anfangswertaufgaben Dissertation", *Universität Hamburg.*

Taubert, K. [1981] "Converging multistep methods for initial value problems involving multivalued maps," *Computing* **27**, 123-136.

Zhou,Y. [2016] *Fractional Evolution Equations and Inclusions: Analysis and Control* (Academic Press, San Diego).

Wiercigroch, M. & de Kraker [2000] *Applied Nonlinear Dynamics and Chaos of Mechanical Systems with Discontinuities* (World Scientific, Singapore).

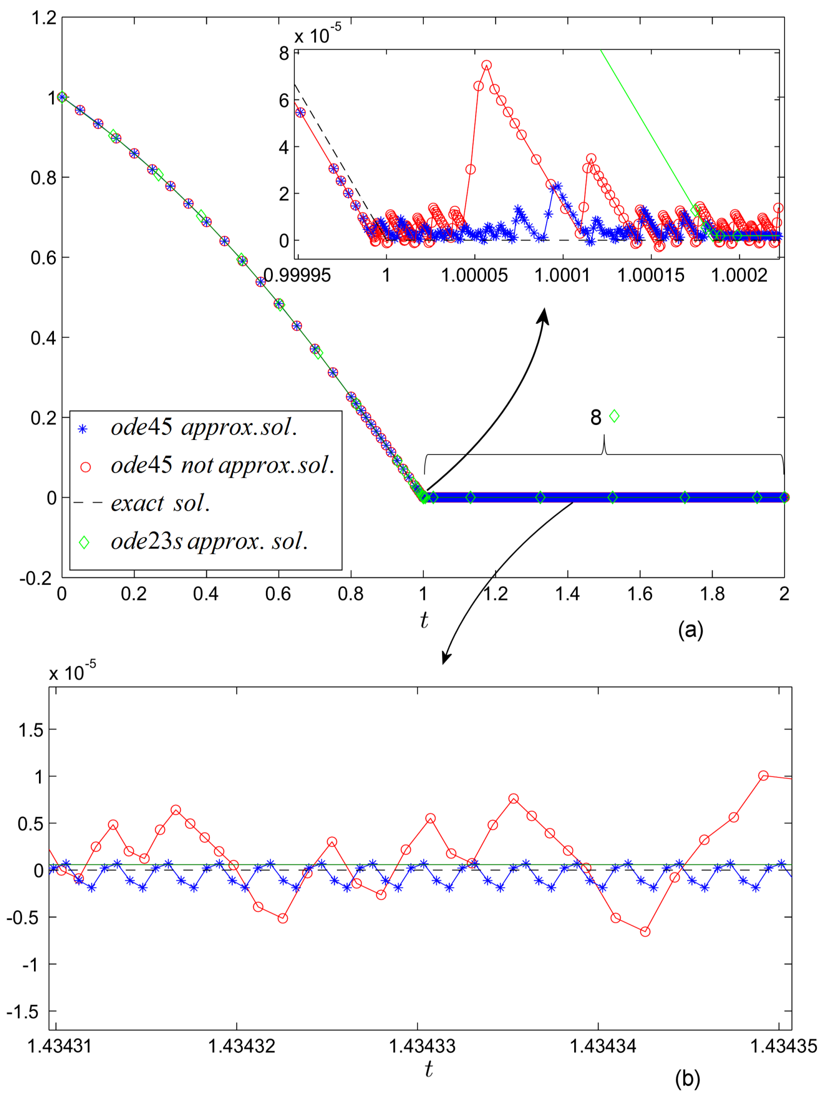

Fig. 3. (a) Graphs of the exact solution of system (6) (interrupted black line), for $t \in [0, 2]$, numerical solution of the non-approximated system (6) obtained with routine *ode*45 (circle red), numerical solution obtained with routine *ode*45 for the approximated system (7) and the numerical solution obtained with routine *ode*23*s* for the approximated system (7), which are overplotted. Upper-right: detail of the sliding beginning at $t = 1$. (b) Detail around the moment $t = 1.434$.

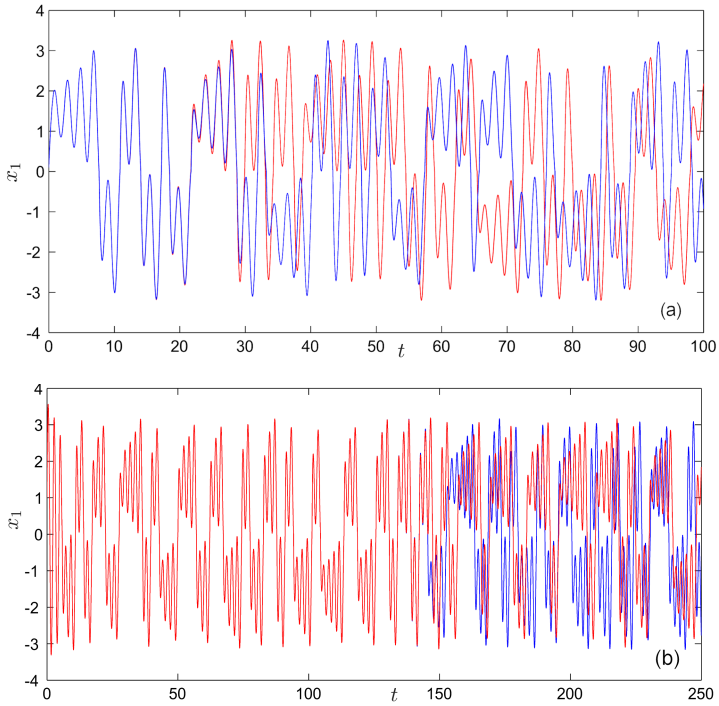

Fig. 4. (a) Graphs of overplotted numerically obtained components x_1 of non-approximated variant of system (8) (red) and of the same system after approximation (blue) (*ode*45 has been utilized). Same initial conditions have been considered. (b) Graphs of overplotted numerically obtained x_1 components of the non-approximated fractional-order variant of system (10) (red) and of the same system after approximation (blue) (ABM method for FDEs has been utilized).