APPROACH OF A CLASS OF DISCONTINUOUS DYNAMICAL SYSTEMS OF FRACTIONAL ORDER: EXISTENCE OF THE SOLUTIONS

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In this letter we are concerned with the possibility to approach the existence of solutions to a class of discontinuous dynamical systems of fractional order. In this purpose, the underlying initial value problem is transformed into a fractional set-valued problem. Next, the Cellina's Theorem is applied leading to a single-valued continuous initial value problem of fractional order. The existence of solutions is assured by a Péano like theorem for ordinary differential equations of fractional order.

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Let us consider a general class of autonomous discontinuous dynamical systems of fractional order modeled by the following Initial Value Problem (IVP)

$$D_*^q x = f\left(x\right) := g\left(x\right) + A\ s(x)\,,\quad x^{(k)}(0) = x_0^{(k)}\quad (k = 0, 1, \ldots, \lceil q \rceil - 1),\ t \in I = [0, \infty), \tag{1}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a function continuous with respect to the state variable, $A = \left(a_{i,j}\right)_{n \asymp n}$ a real constant matrix and s is a piecewise continuous function given by

$$s(x) = \begin{pmatrix} sgn(x_1) \\ \vdots \\ sgn(x_n) \end{pmatrix}.$$

$q \in \mathbb{R}_+$ and D_*^q is considered in this letter as being the most utilized differential operator, the Caputo operator of order q with starting point 0, i.e. (see e.g. [Podlubny, 1999])

$$D_*^q u(t) = \frac{1}{\Gamma(\lceil q \rceil - q)} \int_0^t (t - \tau)^{\lceil q \rceil - q - 1} u^{(\lceil q \rceil)}(\tau) \mathrm{d}\tau.$$

$\Gamma : (0, \infty) \longrightarrow \mathbb{R}$ is the known Euler's Gamma function and $\lceil \cdot \rceil$ denotes the ceiling function that rounds up to the next integer. Thus, $D_*^{\lceil q \rceil}$ is the conventional differential operator of order $\lceil q \rceil \in \mathbb{N}$.

Remark 1. According to the standard mathematical theory [Stefan *et al.*, 1993, §42], we are forced to give the initial conditions for the IVP (1) using fractional derivatives of the function f, or these values are frequently not available. Also, it may not even be clear what their physical meaning is. Therefore, using the Caputo's suggested way, the initial conditions may be specified in the classical way, as in IVP (1).

For the sake of simplicity, we restrict ourselves to the case important for the applications: $q \in (0,1)$ (however the considerations in this paper can be generalized to arbitrary positive q). Therefore, we deal with the following form of IVP (1)

$$D_*^q x = f(x) := g(x) + A\ s(x)\,, \quad x(0) = x_0,\ t \in I = [0, \infty), \tag{2}$$

where we have to specify just one condition since it is easily seen that the number of initial conditions that one needs to specify in order to obtain a unique solution is $\lceil q \rceil = 1$.

Remark 2. The IVP (1) are enough general to include the great generality of systems: for $q = 1$ the systems modeled by the IVP (1) are the known Filippov systems [Filippov, 1988], while for non integer values of q and $A = O_{n\times n}$, the IVP (1) models dynamical systems of fractional order.

The main result of this letter is the following theorem

Theorem 1. *The (1) admits at least one solution.*

For a better readability of the letter, the proof of this theorem shall be given in several steps.

1. Filippov regularization of the right-hand side

The existence and uniqueness of solutions to discontinuous IVPs are essential for discontinuous dynamical systems, because due to the right-hand discontinuity, classical solutions of IVP might not even exist. To provide the existence, it is necessary to modify the right-hand side of IVP (2). For discontinuous vector fields, existence and uniqueness of solutions is not guaranteed in general, no matter what notion of solution is chosen. Also, the classical notion of solution for ordinary differential equations is too restrictive when considering discontinuous vector fields. A possibility to compass this difficulty is to extend the notion of differential equation to differential inclusion, problem solved by Filippov using a generalized concept of solution. Thus, the single valued discontinuous IVP is shifted to the following set-valued one

$$D_*^q x \in F(x),\ x(0) = x_0,\ \text{for almost all } t \in I, \tag{3}$$

where $F : \mathbb{R}^n \Longrightarrow \mathbb{R}^n$ is a set-valued vector function defined on the set of all subsets of $\mathbb{R}^n$. One of simplest definition of F is the following convex form (implicitly used in most introductory references)

$$F(x) = \bigcap_{\varepsilon>0}\ \bigcap_{\mu(M)=0} \overline{conv}\, f\left((x + \varepsilon\, B)\setminus M\right), \tag{4}$$

where, M is the set of discontinuity points of f, B the unit ball in $\mathbb{R}^n$, μ the Lebesgue measure and $\overline{conv}$ the closed convex hull. At the points where the function f is continuous, $F(x)$ will consist of one point, which is the value of f at this point, i.e. $F(x) = \{f(x)\}$. At the discontinuity points, the set $F(x)$ is given by (4). Therefore, $F(x)$ is the convex hull of values of $f(x^*)$, $x^* \in M$, ignoring the behavior on null sets. For example, the Filippov regularization applied to the unidimensional *sign* function leads to the set-valued function

$$Sgn(x) = \begin{cases} \{-1\} & x < 0, \\ [-1,\, 1] & x = 0, \\ \{+1\} & x > 0. \end{cases}$$

and the right-hand side of the IVP (2) becomes

$$F(x) := g(x) + A\ S(x),\ \text{with}\ \ S(x) = \left(Sgn(x_1), \ldots, Sgn(x_n)\right)^T. \tag{5}$$

In order to justify the use of the Filippov regularization in physical systems, ε in (4) must be small enough, so that the motion of the physical system can be arbitrarily close to a certain solution of the differential inclusion.

2. Continuous approximation of the right-hand side

Next, X and Y denote metric spaces (e.g. $\mathbb{R}^n$ as in almost real applications).

Definition 1. A selection of a given set-valued function $F : X \Longrightarrow Y$ is a function $h : X \longrightarrow Y$ satisfying

$$\forall x \in X, \quad h(x) \in F(x).$$

Definition 2. A set-valued function $F : X \longrightarrow Y$ is called upper semicontinuous (u.s.c.) at $x \in X$ if for any neighborhood V of $F(x)$, there exists a neighborhood U of x such that $F(x) \subset V$ for all $x \in V$. F is u.s.c. on X if it is u.s.c. on every point of X.

For practical reasons, it is convenient to characterize a set-valued map $F : X \Longrightarrow Y$ by its graph

$$Graph(F) = \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

Proposition 1. *The set-valued function F defined by (4) is u.s.c. with nonempty closed and convex values.*

Proof. *The proof can be found e.g. in [Aubin & Cellina, 1984, p. 102].* ■

Remark 3. Due to the symmetric interpretation of a set-valued map as a graph (see e.g. [Aubin & Frankowska, 1990]) we shall say that a set-valued map satisfies a property if and only if its graph satisfies it. For instance, a set-valued map is said to be convex if and only if its graph is a convex set.

The following known theorem (Cellina's Theorem or "Approximative Selection Theorem") will be a main tool used in the proof of Theorem 1.

Theorem 2. *[Aubin & Cellina, 1984; Aubin & Frankowska, 1990] Let $F : X \longrightarrow Y$ be upper u.s.c. set-valued with Y a Banach space. If the values of F are nonempty and convex then, for every $\varepsilon > 0$, there exists a locally Lipschity function $f_\varepsilon : X \longrightarrow Y$ such that*

$$Graph(f_\varepsilon) \subset Graph(F) + \varepsilon B.$$

The proof of Theorem 2 is constructive (see the mentioned reference) in that it provides a method to explicitly construct the selection f_ε with ε parameter. This is an important advantage which allows to find practical approximations in real applications.

3. Existence of the solutions for fractional equations

Let us consider the following IVP of fractional order with $q \in (0, 1)$

$$D_*^q x = f(x), \; x(0) = x_0, \tag{6}$$

with $x \in \mathbb{R}^n$. The following existence result corresponds to the classical Péano existence theorem for first order equations

Theorem 3. *[Kai, 2010] Assume f in IVP (6) is continuous and bounded. Then there exists a solution to IVP (6).*

Proof. see [Kai, 2010, Corollary 6.4 p. 92]. ■

Next we can give

Proof. [Proof of Theorem 1] Applying Filippov regularization, the right hand side of (1) transforms into the set-valued function F given by (5). F is a convex u.s.c. (Proposition 1) and non-empty valued function (Remark 3). Therefore, Theorem 2 can be used and the IVP (1) becomes a continuous IVP of fractional order to which Theorem 3 applies and the proof is complete. ■

Remark 4. i) Using the Fillipov regularization, the single valued IVP may be considered as transforming into a set-valued IVP of fractional order. However, for practical purposes, fractional single-valued IVPs are more accessible by numerical point of view, since there are several ways to approximate their solutions.
ii) The uniqueness for general fractional equations is treated in [Kai, 2010, §6.2] the underlying theorem corresponding to the well-known Picard- Lindelöf Theorem for equations of integer order and is based on Lipschitz continuity. However, for our case of systems, the uniqueness is checked since from Theorem 2 the approximate selection is locally Lipschitz continuous.

For example, let us consider the following fractional discontinuous variant of the Chua's system

$$\begin{aligned} D_*^q x_1 &= -2.57x_1 + 9\,x_2 + 3.87\ sgn\,(x_1), \\ D_*^q x_2 &= x_1 - x_2 + x_3, \\ D_*^q x_3 &= -p\,x_2, \end{aligned} \tag{7}$$

where p is a real parameter. The graph of the first component of the right-hand side of (7), the scalar function $f_1\,(x_1, x_2) = -2.57x_1 + 9\,x_2 + 3.87\ sgn\,(x_1)$, is plotted in Fig. 1(a). The convex hull of $f_1\,(0, x_2)$ is the dashed region and represents the graph of the set-valued function $F_1(0, x_2)$. The underlying set-valued IVP is

$$\begin{aligned} D_*^q x_1 &\in -2.57x_1 + 9\,x_2 + 3.87\ Sgn\,(x_1), \\ D_*^q x_2 &= x_1 - x_2 + x_3, \\ D_*^q x_3 &= -p\,x_2. \end{aligned} \tag{8}$$

A possible approximation (selection) of F_1 in the neighborhood of $(0, x_2)$ can be, for example, a smooth cubic polynomial function $h\,(x_1, x_2) = ax_1^3 + bx_1^2 + cx_1 + 9x_2$ [Danca & Codreanu, 2002] where a, b, d and d have to be determined from the continuity and differentiability conditions in $\pm\varepsilon$, $x_2 \in \mathbb{R}$ (Fig. 1(b). Next, the system can be either numerically integrated using e.g. the fractional Adams–Bashforth–Moulton method discussed in [Diethelm *et al*, 2002], or approximated using some frequency-domain method *(*see e.g. [Charef *et al*, 1992]). More approximations ways for fractional operators can be found e.g. in [Vinagre *et al.*, 2002].

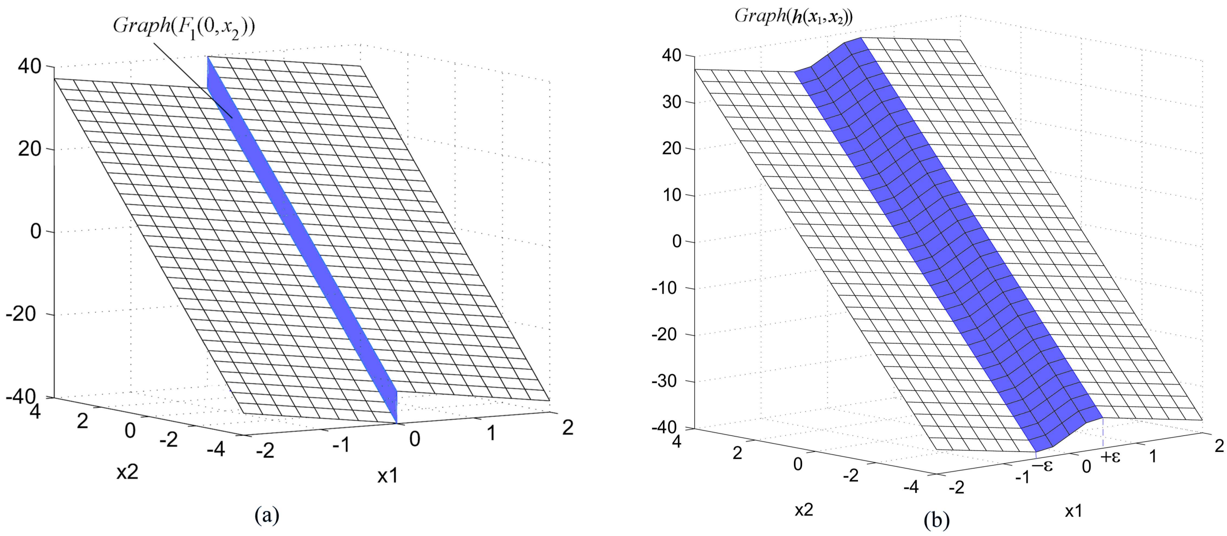

Fig. 1. a) Graph of $F_1(0, x_2)$; b) Approximation of F_1.

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