Fractional-order attractors synthesis via parameter switchings

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Abstract

In this paper we provide numerical evidence, via graphics generated with the help of computer simulations, that switching the control parameter of a dynamical system belonging to a class of fractional-order systems in a deterministic way, one obtains an attractor which belongs to the class of all admissible attractors of the considered system. For this purpose, while a multistep numerical method for fractional-order differential equations approximates the solution to the mathematical model, the control parameter is switched periodically every few integration steps. The switch is made inside of a considered set of admissible parameter values. Moreover, the synthesized attractor matches the attractor obtained with the control parameter replaced with the averaged switched parameter values. The results are verified in this paper on a representative system, the fractional-order Lü system. In this way we were able to extend the applicability of the algorithm presented in earlier papers using a numerical method for fractional differential equations.

1 Introduction

In this paper we investigate numerically the possibility to synthesize any attractor of a class of fractional-order dynamical systems depending on a real parameter by periodic parameter switchings.

While in [1, 2] the attractors synthesis has been achieved and tested for a wide class of continuous dynamical systems of integer order, in the present paper we numerically check that this algorithm can be adapted to a general class of dynamical systems of fractional order. For this purpose we have chosen the Lü system of fractional order 0.9 as a prototypical example. This value of the order has been chosen because results presented in earlier papers indicate that orders close to 1 exhibit practically more relevant phenomena. In particular, chaotic behaviour tends to disappear if the order becomes too small [3]. Via a numerical method for differential equations of fractional order described in more detail below, our algorithm allows to synthesize any attractor of the fractional Lü system by switching the control parameter within a properly chosen set of values following some deterministic or even random rule.

As it is known, a given dynamical system depending on a real parameter may have, for a fixed parameter value, several attractors, called *local attractors* (see e.g. [4, 5, 6]). These invariant sets form the so called *global attractor* [6]. Given a value of the control parameter, each local attractor can be reached, subject to an appropriate choice of the of the initial conditions. In other words, the global attractor contains all solutions, including stationary solutions, periodic solutions, as well as chaotic attractors, relevant to the asymptotic behaviors of the system. The study of global attractors is one of the major research topics in dynamical systems, in particular within the context of PDEs (see e.g. [7]).

Because of the numerical character of this paper, for convenience, we omit the attribute "global" unless necessary and, by a slight abuse of notation, by *attractor* we will understand the working notion of ω-*limit set* approximation [8] obtained for a fixed, well determined initial condition, after a sufficiently long period of transients neglected. Thus, the attractor is understood sometimes as being an invariant set that attracts its neighborhood, or attraction basin, and equals the ω-limit set of one of its neighborhoods. (A discussion of the concept of an attractor in the theory of finite-dimensional dynamical systems is given by Milnor [9]).

Therefore, in our numerical experiments, we deal with orbits obtained by some numerical method paying a special attention to initial conditions. For this purpose, all the numerical experiments for a specific case were made considering the same attraction basin for the initial conditions.

Nowadays, it has been found that in interdisciplinary fields, there are systems which can be described by fractional differential equations much more efficiently than by classical techniques. The main reason for using the integer-order models was the absence of solution methods for fractional differential equations while one possible explanation of such unpopularity could be that there are multiple nonequivalent definitions of fractional derivatives [10, 11]. Also, the fractional derivatives have no evident geometrical interpretation because of their nonlocal

character. However, during the last 10 years, fractional-order dynamical systems have started to attract the attention of the scientific community.

Fractional mathematical concepts allow to describe certain real objects more accurately than the classical "integer" methods. Examples of such real objects that can be elegantly described with the help of fractional derivatives displaying fractional-order dynamics may be found in many fields of science like physics, engineering, mathematical biology, medicine, finance etc. (see e.g. [12, 13, 14, 15, 16, 17, 18, 19, 20]). Therefore the extension of the synthesis algorithm to systems of this kind is of real interest.

The paper is structured as follows: Section 2 presents some basic concepts, in Section 3 the attractors synthesis (AS) algorithm is explained, while in Section 4 the algorithm is applied to the particular case of a Lü system of fractional-order. The last section presents the conclusions.

2   Notions and preliminaries

The attractors synthesis (AS) algorithm, introduced by Danca et al. in [1], can be applied to a large class of integer-order autonomous dynamical systems modeled by the initial value problem (IVP)

$$S: \ \dot{x} = f_p(x), \quad x(0) = x_0, \tag{1}$$

where f_p is an $\mathbb{R}^n$-valued function with a single bifurcation parameter $p \in \mathbb{R}$, $n \geq 3$, having the expression

$$f_p(x) = g(x) + pAx, \tag{2}$$

with $g : \mathbb{R}^n \to \mathbb{R}^n$ a continuous nonlinear function, A a real constant $n \times n$ matrix, $x_0 \in \mathbb{R}^n$, and $t \in I = [0, \infty)$. This class of dynamical systems is rather large; it contains well known systems such as Lorenz, Rössler, Chen, Chua, Lü, Rikitake, Lotka-Volterra, Fabrikant-Rabinovich etc.

In this paper we extend the applicability of AS algorithm to a class of fractional-order dynamical systems modeled by the IVP

$$S: D_*^q x = f_p(x), \quad x^{(k)}(0) = x_0^{(k)} \quad (k = 0, 1, \dots, \lceil q \rceil - 1), \tag{3}$$

where f_p has the form (2), q is some positive real number and D_*^q denotes the Caputo differential operator of order q with starting point 0, i.e.

$$D_*^q x(t) = \frac{1}{\Gamma(\lceil q \rceil - q)} \int_0^t (t - \tau)^{\lceil q \rceil - q - 1} D^{\lceil q \rceil} x(\tau) d\tau,$$

(see e.g. [10, 21]). Here $\lceil \cdot \rceil$ denotes the ceiling function that rounds up to the next integer, and $D^{\lceil q \rceil}$ is the standard differential operator of order $\lceil q \rceil \in \mathbb{N}$.

Throughout this paper, we shall assume the existence and uniqueness of solutions on the maximal existence interval I.

To implement the AS algorithm, it is necessary to provide a numerical scheme for the solution of the IVP (3). To this end we use the fractional Adams-Bashforth-Moulton (ABM) method discussed in [21] and analyzed in a detailed way in [22]. Specifically, the algorithm works by introducing a discretization with grid points $t_i = hi$, $i = 0, 1, \ldots$, with a presassigned step size h. For $i = 0, 1, 2, \ldots$ it then first computes a preliminary approximation x_{i+1}^{P} for $x(t_i)$ via the formula

$$x_{i+1}^{\mathrm{P}} = \sum_{j=0}^{\lceil q \rceil - 1} \frac{t_{i+1}^j}{j!} x_0^{(j)} + \frac{1}{\Gamma(q)} \sum_{j=0}^{i} b_{j,i+1} f_p(x_j) \tag{4}$$

where

$$b_{j,i+1} = \frac{h^q}{q} \left((i+1-j)^q - (i-j)^q \right), \tag{5}$$

and then it determines the actual final approximation x_{i+1} for $x(t_{i+1})$ that is effectively used via

$$x_{i+1} = \sum_{j=0}^{\lceil q \rceil - 1} \frac{t_{i+1}^j}{j!} x_0^{(j)} + \frac{h^q}{\Gamma(q+2)} \left(\sum_{j=0}^{i} a_{j,i+1} f_p(x_j) + f_p(x_{i+1}^{\mathrm{P}}) \right) \tag{6}$$

with

$$a_{0,i+1} = i^{q+1} - (i-q)(i+1)^q \tag{7}$$

and

$$a_{j,i+1} = (i-j+2)^{q+1} + (i-j)^{q+1} - 2(i-j+1)^{q+1} \tag{8}$$

for $j = 1, 2, \ldots, i$. For a detailed derivation of these formulas we refer to [21]; a deep mathematical analysis of this approach is given in [22].

Our preference for the fractional ABM scheme in comparison to the large number of other possibilities that can be found in the literature is due to the observations of Tavazoei et al. [23, 24] who have found that many other methods tend to be unreliable when used to numerically determine whether a fractional system is stable or not.

The Gamma function was approximated in this work using a variant of the so-called Lanczos approximation [25]

$$\Gamma(z) = \frac{\sum_{i=0}^{6} p_i z^i}{\prod_{i=0}^{6} (z+i)} (z+5.5)^{z+0.5} \mathrm{e}^{-(z+5.5)}$$

for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ where the coefficients p_i are shown in Table 1.

Compared to the standard methods for ODEs of integer order, where the current approximation x_k depends only on the results of a few backward steps, the fractional ABM scheme (like all reasonable numerical methods for fractional differential equations) requires the entire backward integration history at each point in time. In other words, each current calculated value depends on all previous values $x_0, x_1, \ldots, x_{k-1}$. This feature implies a serious drawback with respect to the required computing time but it is necessary to appropriately reflect the memory effects possessed by fractional differential operators.

Table 1: Coefficients of the Lanczos approximation of the Gamma function.

i	p_i
0	75122.6331530
1	80916.6278952
2	36308.2951477
3	8687.2452971
4	1168.9264948
5	83.8676043
6	2.5066283

Remark 1 *Another way to deal with fractional derivatives, suitable for Matlab programming, uses of the frequency domain approximation and was proposed by researchers on automatic control (see e.g.[19, 26, 27]).*

3 The AS algorithm

In order to describe the attractors synthesis algorithm, we shall use the following notation:

- $\mathcal{A}$ — *the set of all global attractors depending on parameter p, including attractive stable fixed points, limit cycles and chaotic attractors;*

- $\mathcal{P} \subset \mathbb{R}$ — *the set of the corresponding admissible values of p;*

- $\mathcal{P}_N = \{p_1, p_2, \ldots, p_N\} \subset \mathcal{P}$ — *a finite ordered subset of $\mathcal{P}$ containing N different values of p;*

- $\mathcal{A}_N = \{A_{p_1}, A_{p_2}, \ldots, A_{p_N}\} \subset \mathcal{A}$ — *the set of global attractors corresponding to $\mathcal{P}_N$ (see Remark 2);*

- $I = \bigcup_{j=1,2,\ldots} \left(\bigcup_{i=1}^{N} I_{ij} \right)$ *with the time-subintervals I_{ij} of time length Δt_i, $i = 1, 2, \ldots, N$ for all $j = 1, 2, \ldots$.*

The structure of I_{ij} will be defined later in this section.

It is assumed that all the values of $\mathcal{P}_N = \{p_1, p_2, \ldots, p_N\}$ for which the system has regular or chaotic motion are accessible.

Remark 2 *$\mathcal{A}$ is non-empty and it follows naturally that for the considered class of systems, a bijection between the sets $\mathcal{P}$ and $\mathcal{A}$ may be defined. Thus, to any $p \in \mathcal{P}$, there is a unique corresponding attractor and vice versa.*

The AS algorithm consists in switching p as a periodic piecewise continuous function $p : I \to \mathcal{P}_N$ of period $T = \sum_{i=1}^{N} \Delta t_i$ while the IVP (1-2) is integrated, i.e.

$$p(t) = p_{\varphi(i)} \text{ for } t \in I_{ij}, \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots,$$

where φ permutes the subset $\{1, 2, \ldots, N\}$.

Via numerical computations and graphical simulations we proved in [1] that the AS method can force the system S to evolve either along any stable (regular) attractor or on any unstable (chaotic) attractor, whatever the initial behavior was and moreover, we verified that the obtained attractor belongs to the set of all admissible attractors of S.

Using a numerical method with fixed step size h to integrate IVP (1)-(2), the simplest way to implement the AS algorithm numerically is to choose Δt_i as multiple of the step size h, $\Delta t_i = m_i h$, where the "weights" m_i are some positive integers to be chosen empirically. Schematically, for a fixed value of h under consideration, the AS method can be written simplified as the "time evolution" (trajectory)

$$[m_1 p_{\varphi(1)}, \; m_2 p_{\varphi(2)}, \ldots, m_N \; p_{\varphi(N)}], \tag{9}$$

with the following meaning: for the first m_1 integration steps, $p = p_{\varphi(1)}$, for the next m_2 integration steps, $p = p_{\varphi(2)}$ and so on until the N-th time subinterval of length $m_N h$ where $p = p_{\varphi(N)}$ for m_N times, after which the algorithm repeats on the next N time subintervals and so on.

For example, by the scheme $[1p_2, 3p_1]$ where $\varphi(1) = 2$, $\varphi(2) = 1$, $m_1 = 1$ and $m_2 = 3$, one should understand the sequence of $p : p_2, 3p_1, p_2, 3p_1, p_2, 3p_1, \ldots$ which means that while the considered numerical method integrates (1-2), with fixed step size h, p switches in each time subinterval I_{ij}, between the values of $\mathcal{P}_2 = \{p_1, p_2\}$.

Here and in the following, let A^* denote the *synthesized attractor* generated by the AS method and A_{p^*} the *averaged attractor* with

$$p^* = \frac{\sum_{k=1}^{N} m_k p_{\varphi(k)}}{\sum_{k=1}^{N} m_k}, \tag{10}$$

obtained by integrating the IVP (1-2) for $p = p^*$.

Numerical evidence on specific examples indicates that A^* belongs to $\mathcal{A}$ and, moreover, A^* is "identical" with A_{p^*}. By attractors "identity" we understand here a best possible overlap of orbits in the phase space, and a small Hausdorff distance for the stable fixed points and limit cycles. (In the case of chaotic attractors this notion should be considered taking into account that the a chaotic attractor is entirely generated after an infinite time.) Thus, to convince ourselves that $A^* \in \mathcal{A}$, we computed A^* and compared it successfully with A_{p^*} with p^* given by (10).

If we denote $\alpha_k = m_k / \sum_{k=1}^{N} m_k$, it is easy to see that p^* is a convex combination: $p^* = \sum_{k=1}^{N} \alpha_k p_{\varphi(k)}$ because $\sum_{k=1}^{N} \alpha_k = 1$. Therefore $p^* \in \mathcal{P}$. Taking into account the bijection between $\mathcal{P}$ and $\mathcal{A}$, we are entitled to consider that the same convex structure is preserved into $\mathcal{A}$. Thus $A^* \in \mathcal{A}$.

Of course, since there are infinitely many ways to choose the weights m_k in (10), for any given p^* and N there are infinitely many possible choices of $\mathcal{P}_N$ that satisfy this equation.

Because of the mentioned convexity property, the AS algorithm can be applied not only in the deterministic way (9) but even in some random way [2].

As shown by the numerical experiments of this paper and those in [1] and [2], the AS algorithm proves to be computationally and numerically robust with respect to parameter switching.

4 Attractors synthesis of the fractional Lü system

The AS algorithm was successfully tested on integer-order chaotic dynamical systems. In this section we show numerically that it works for fractional chaotic dynamical systems too. For this purpose we choose the fractional variant of the system found in 2002 by Lü et al. [28] which unifies the Lorenz and Chen systems (see the proof in [28])

$$
\begin{aligned}
D_*^q x_1 &= p(x_2 - x_1), \\
D_*^q x_2 &= -x_1 x_3 + 28 x_2, \\
D_*^q x_3 &= x_1 x_2 - 3 x_3.
\end{aligned}
\tag{11}
$$

Because for many real fractional systems the order of the fractional differential operators is less than 1, we fix in this paper $q = 0.9$ which is a typical value that exhibits all the relevant phenomena (the development of the dynamics of this system as q varies is well known [3]). The real (mathematical) order of fractional systems is the sum of the orders of all involved derivatives. As it is well known, chaos cannot occur in autonomous continuous-time systems of integer-order less than three according to the Poincaré–Bendixon theorem. In autonomous fractional-order systems, like the Lü system, this is not the case as one can be seen in this paper, where for $q = 0.9$ the real order of the system is 2.7.

In order to apply the AS algorithm, the Lü system (11) was integrated with the fractional ABM method with 15000 steps of step size $h = 0.005$.

It is easily seen that the number of initial conditions that one needs to specify in our case $0 < q \leq 1$ is $\lceil q \rceil$, i.e. just one condition. Initial conditions are generally not critical (except for the case when several local attractors coexist for a specific p value). Therefore, A^* and A_{p^*} for this example are generated numerically starting from the same initial conditions, here $(-0.5, 0.5, 0.5)$. Also, as stated in Section 3, transients were neglected, especially for stable limit cycles.

The coincidence between A^* and A_{p^*} was verified numerically by means of phase plots and Hausdorff distance [29] for the stable limit cycles, which was of order of 10^{-1}. However for the integer-order systems, Hausdorff distance is smaller (up to the order of 10^{-3}). This could be possible because of the smaller errors of the numerical methods for integer-order differential equations (compare, e.g., the well known $O(h^2)$ error bound for our one-step Adams-Bashforth-Moulton method in the case $q = 1$ with the weaker bounds for the fractional version discussed in detail in [22]).

To explore the Lü system dynamics, the bifurcation diagram was drawn (Figure 1).

First, let us consider the scheme (9) for $N = 2$ and weights $m_1 = m_2 = 1$, i.e. $[1p_1, 1p_2]$ for $p_1 = 33.5$ and $p_2 = 35.5$. For these values, (10) gives the value $p^* = (1 \cdot p_1 + 1 \cdot p_2)/2 = 34.5$ which belongs inside of a periodic window in the parametric space. The synthesized attractor A^* is a stable limit cycle (Figure 2), even though A_{p_1} and A_{p_2} are chaotic attractors (see also Fig. 1). As can be seen from Fig. 2 c, A^* and A_{p^*} are identical.

A chaotic attractor can be synthesized with $p_1 = 38.8$ and $p_2 = 55$. Using the scheme $[1p_1, 2p_2]$, $p^* = 49.6$ and the synthesized chaotic attractor A^* coincides with $A_{49.6}$ (see Fig. 3). The non-uniqueness of solutions for (10) can be underlined if the same attractor $A_{34.5}$, synthesized above with the scheme $[1p_1, 1p_2]$, is generated with another scheme, for example with $N = 3$: $[1p_1, 1p_2, 1p_3]$ for $p_1 = 32$, $p_2 = 35$, $p_3 = 36.5$ (Fig. 4).

The dynamics of the synthesized attractor A^* does not depend on the dynamics of the underlying attractors. For example, a chaotic attractor can be synthesized starting from two different kinds of attractors like those used in Fig. 3, where A_{p_1} is chaotic and A_{p_2} has a stable fixed point, but can be synthesized starting from e.g. two regular motions withe the scheme $[1p_1, 1p_2]$ for $p_1 = 32$ and $p_2 = 34.5$, when A^* is a chaotic attractor identical with $A_{33.25}$ (Fig. 5).

5   Conclusion

In this paper we demonstrated numerically, on the particular case of the fractional-order Lü system, that switching deterministically the control parameter p inside of a set of parameter values, the obtained (synthesized) attractor A^* belongs to the set of all attractors of this system and, moreover, is identical to the attractor obtained for $p = p^*$ with p^* given by (10). Thus, the AS algorithm seems to work not only for integer-order dynamical systems (as proved numerically in [1] and [2]), but for a class of fractional-order dynamical systems modeled by the IVP (3).

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Figure 1: Bifurcation of the the Lü system. 5000 integration steps were utilized for each p value in order to stabilize the values of x_1.
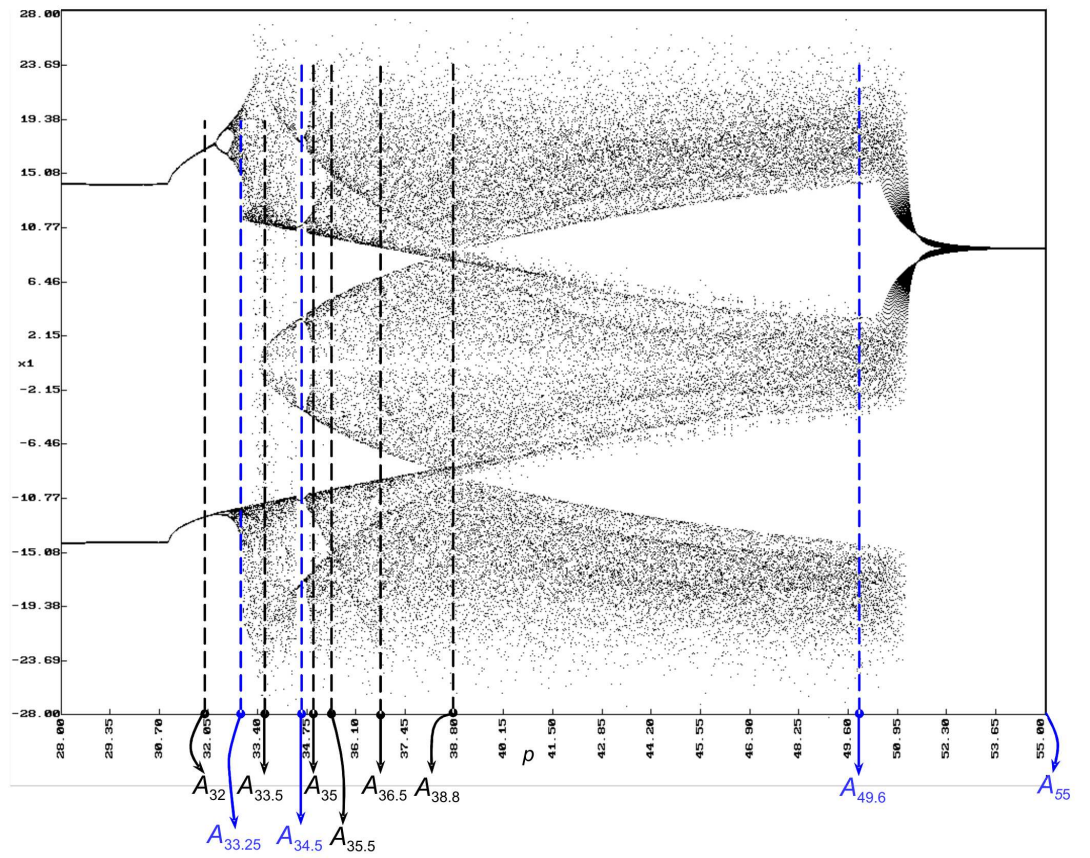
Figure 2: AS algorithm $[1p_1, 1p_2]$ applied to the Lü system for $p_1 = 33.5$ and $p_2 = 35.5$; a) The attractor $A_{33.5}$; b) The attractor $A_{35.5}$; c) The attractors A^* and A_{p^*} superimposed.
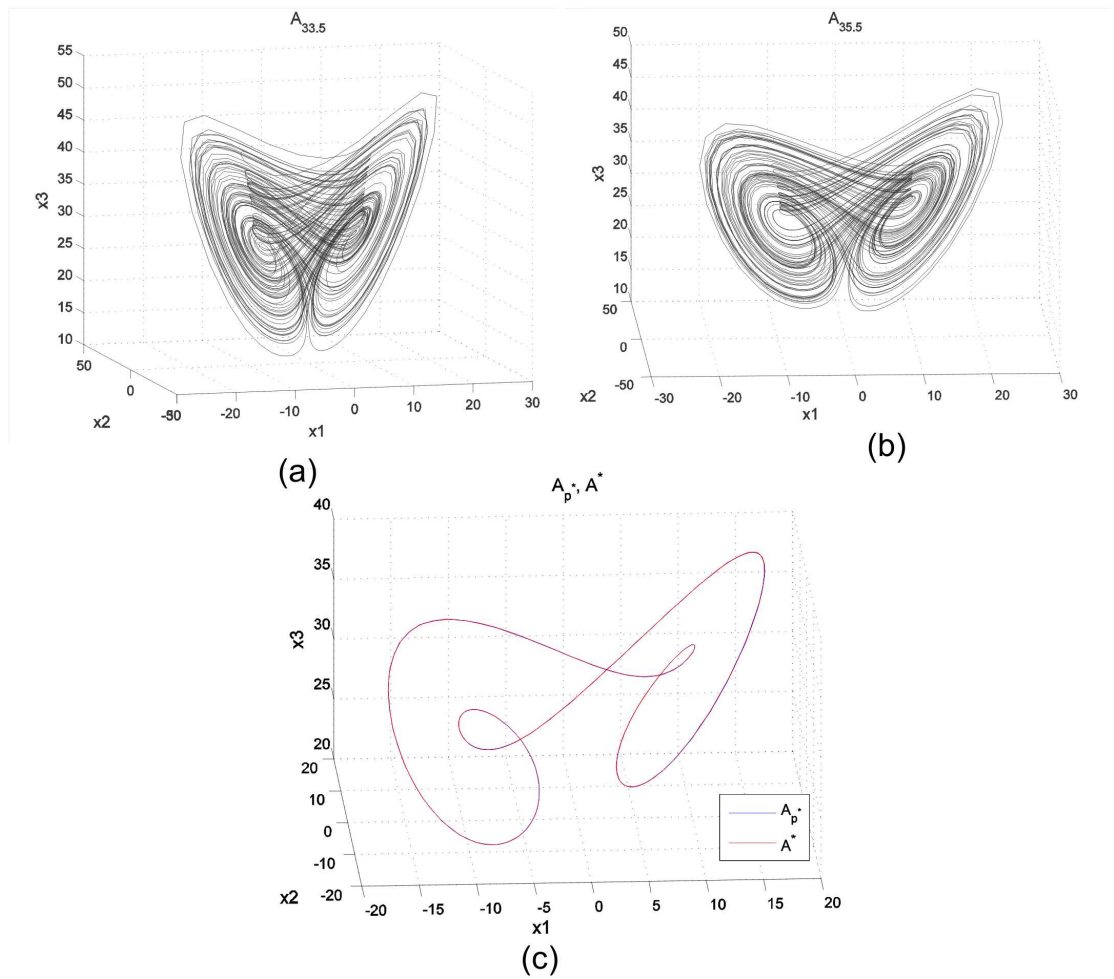

Figure 2

Figure 3: AS algorithm $[1p_1, 2p_2]$ with $p_1 = 38.8$ and $p_2 = 55$. a) The attractor $A_{38.8}$; b) The attractor A_{55}; c) The attractors A^* and A_{p^*} superimposed.
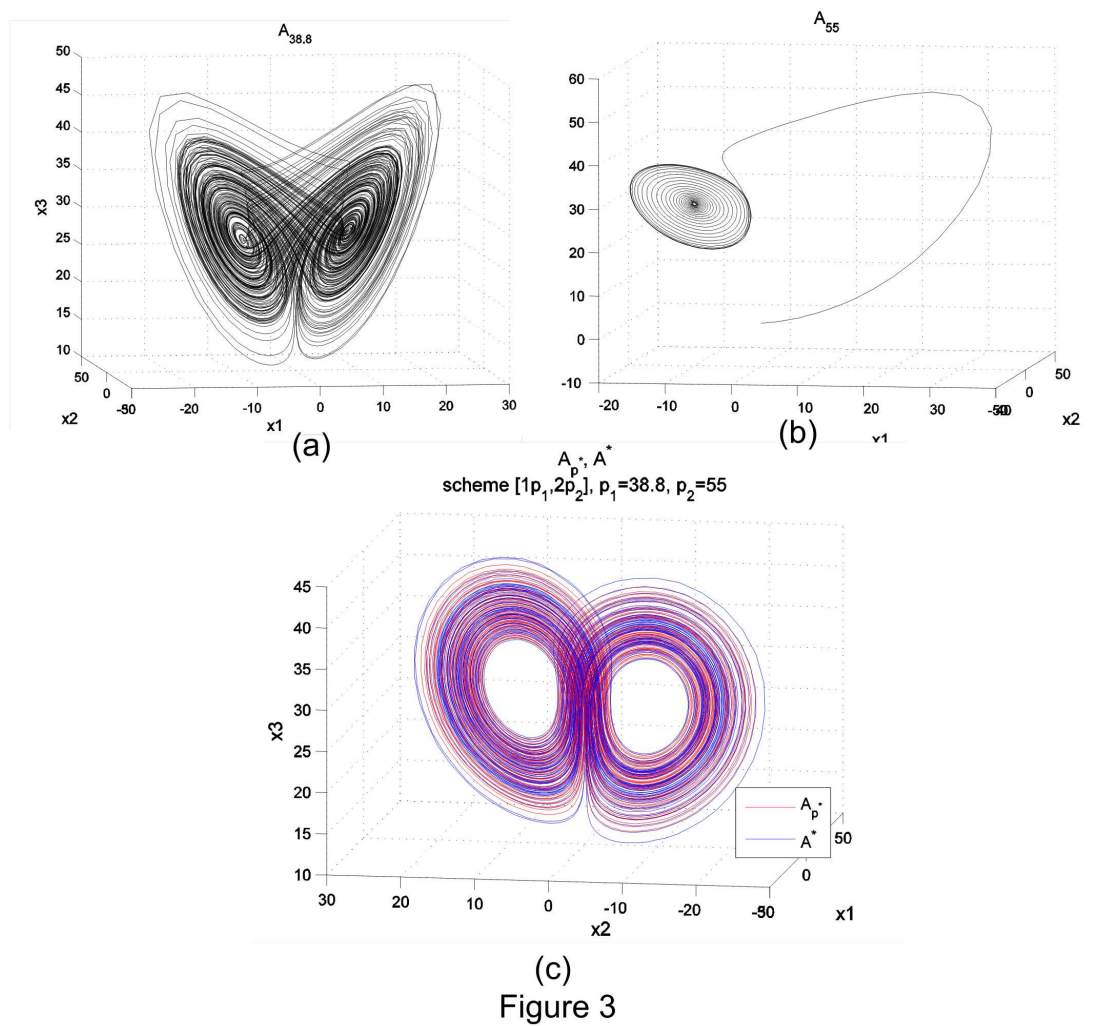

(a)

(b)

(c)

Figure 3

Figure 4: AS algorithm $[1p_1, 1p_2, 1p_3]$ for $p_1 = 32$, $p_2 = 35$, $p_3 = 36.5$. a) The attractor A_{32}; b) The attractor A_{35}; c) The attractor $A_{36.5}$; d) The attractors A^* and A_{p^*} superimposed.
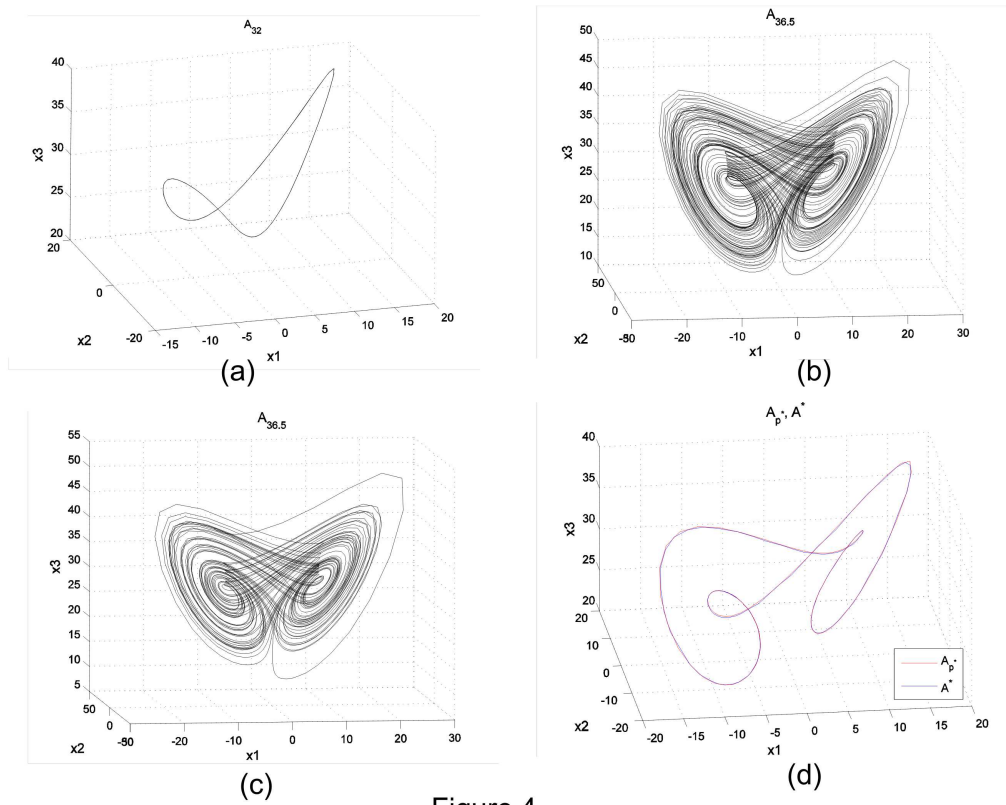

Figure 4

Figure 5: AS algorithm $[1p_1, 1p_2]$ with $p_1 = 32$ and $p_2 = 34.5$. a) The attractor A_{32}; b) The attractor $A_{34.5}$; c) The attractors A^* and A_{p^*} superimposed.
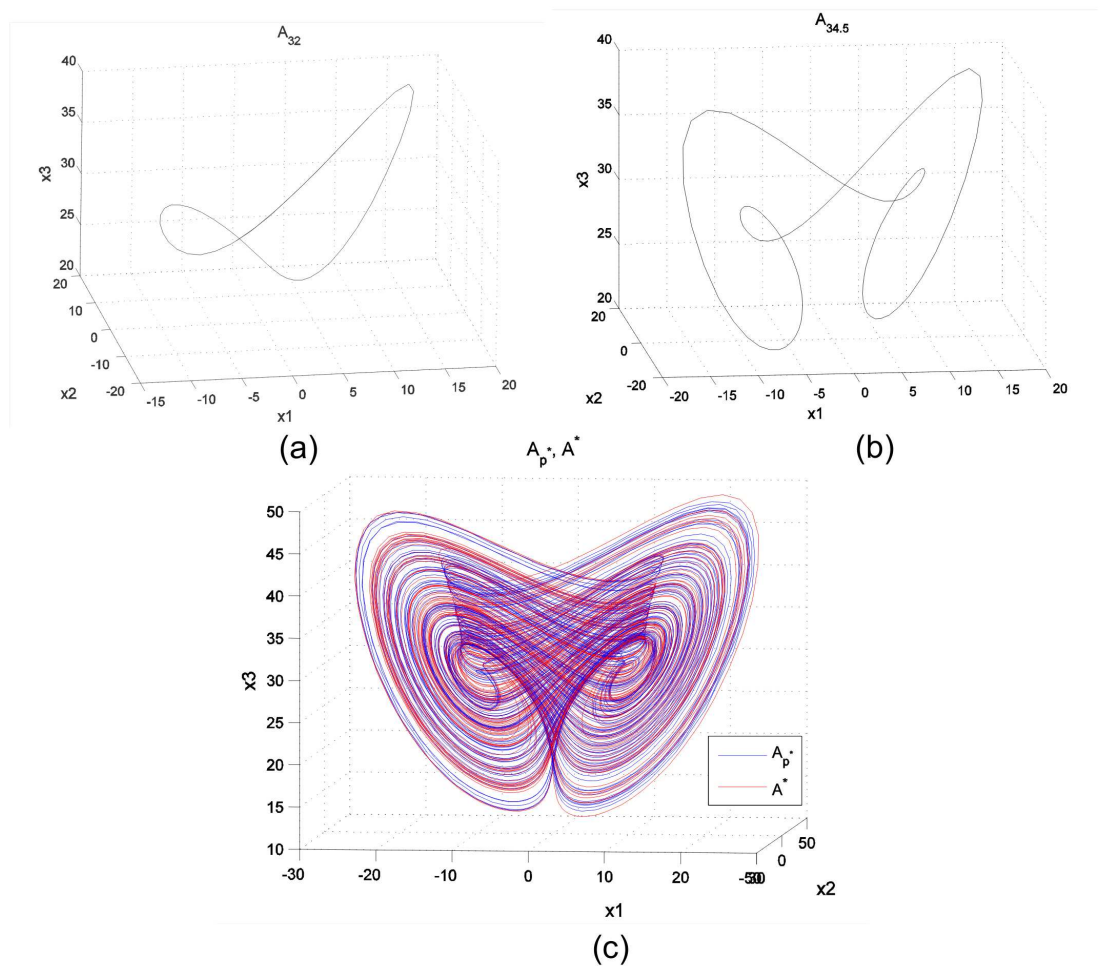

(a)

(b)

(c)

Figure 5

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