Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 837262, 23 pages
doi:10.1155/2011/837262

Research Article

Calculation of the Structure of a Shrub in the Mandelbrot Set

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Received 26 October 2010; Revised 19 January 2011; Accepted 1 March 2011

Academic Editor: Recai Kilic

We calculate the external arguments of the structure of any shrub in the Mandelbrot set. Before calculating, we revise, expand, and clarify some tools useful for this paper: harmonics, pseudoharmonics, the concept of structure, the structure of a shrub, and the ancestral route. Finally we present the main contribution of this paper, a three-step algorithm which allows us to calculate the structure of the shrub. In the first step, we use pseudoharmonics that were previously introduced by us, in order to calculate the first and last external arguments of a structural node. In the second step, starting from two general properties of the Misiurewicz points external arguments introduced here by us, we present a new method to calculate the intermediate external arguments. In the last step we introduce a third property that allows us to calculate the external arguments of the representatives of the branches emerging from the structural nodes.

1. Introduction

Although the Mandelbrot set [1, 2] was discovered in 1980, such is the fascination that continues exerting in the scientific world that it still remains heavily studied. Since this set is a mathematical body, mathematicians are those who have contributed most to its study [3–6]. However, we should not forget that the Mandelbrot set is the most representative paradigm of what we mean by chaos, and experimental scientists are strongly interested in chaotic phenomena. And these scientists, among which we find ourselves, analyze the Mandelbrot set with a different look, usually through graphical and computational analysis, to help them explain the chaotic behaviour found in their experiments. This is the case of this paper, based on a very large number of computational experimental data, which are analyzed computationally and in many cases with the help of graphic tools.

Let us see that, indeed, the study of the Mandelbrot set is now in full force. Some authors have introduced the fruitful field of generalized Mandelbrot set [7–15], and some of our previous papers have been generalized and improved by these authors. In the same way, we think that this paper is susceptible of being applied to a generalized M-set. Another very interesting field of application in which other authors work is the perturbation of the Mandelbrot set, for example by introducing noise [8, 12, 14–22].

Due to the enormous complexity of the Mandelbrot set, in previous papers we have made an effort to visualize its ordering and to simplify as much as possible its study. To visualize its ordering we have introduced the shrubs [23], and to simplify the study we do not analyse the whole Mandelbrot set but only what we call structure [24, 25].

In Section 2, we review these and other topics needed to calculate the structure of any shrub in the Mandelbrot set. We start with harmonics and pseudoharmonics that will be used later, then, we see and clarify with new contributions the concept of structure of the Mandelbrot set, next, we introduce shrubs and analyze their structure, and, finally, we introduce the ancestral route. Section 2 is essentially a reminder with few new contributions, and therefore this section can be skipped by readers already familiar with these issues, and it is in Section 3 where we give almost all the contributions of this paper, as will be seen later.

The most important elements of the Mandelbrot set are hyperbolic components (HCs) [5] and Misiurewicz points [26–28]. As mentioned before, we only treat the structure of the set, which is as its skeleton. We denominate structural HCs to the HCs of the structure, and in the same way we denominate structural Misiurewicz points to the Misiurewicz points of the structure. Moreover, external arguments (EAs) of Douady and Hubbard [29] are the best way to identify both the HCs, which are periodic, and the Misiurewicz points, which are preperiodic.

In this paper we calculate the structure of any shrub in the Mandelbrot set. That is, we will calculate the EAs of structural HCs and structural Misiurewicz points of the shrub we are considering, based solely on the EAs of the generator of the shrub. Let us see next the background in this field.

Devaney and Moreno-Rocha [30] calculate for the first time the EAs of a Misiurewicz point, namely, the main node of a primary shrub. In [31], by assuming known the EAs of the representatives of the structural branches, we calculate the EAs of any structural node of any shrub by using pseudoharmonics and pseudoantiharmonics. In the present paper, pseudoharmonics will also be used, and therefore they will be reviewed in Section 2.1.2.

In Section 3 we give a three-step algorithm to calculate the structure of any shrub in the Mandelbrot set, which is the aim of this paper. In the first step, using pseudoharmonics that were previously introduced by us [31], we calculate the first and last external arguments of a structural node. This first step does not involve any new contribution since the first and last external arguments can be calculated in some particular cases in [30], and in general in our work [31]. In the second step, we introduce two general properties of the external arguments of the Misiurewicz points in order to introduce a new method to calculate the intermediate external arguments of a Misiurewicz point. In the third and final step we introduce a third property that allows us to calculate the external arguments of the representatives of the structural branches emerging from the structural nodes.

Based on the three-step algorithm we calculate, as we said above, the EAs of all structural elements of any shrub starting exclusively from the EAs of the generator of the shrub. This generator, as we will see, is always known.

2. Previous Considerations

2.1. Harmonics and Pseudoharmonics

Harmonics/antiharmonics and pseudoharmonics/pseudoantiharmonics can be seen in [31, 32]. Since here we are going to use only harmonics and pseudoharmonics, we will only see them. We begin by introducing the harmonics of the pair of EAs of an HC (from now on, for our convenience, EAs will be given only in the binary expansion form).

2.1.1. Harmonics

Let $(.\overline{a_1}, .\overline{a_2})$ be the pair of EAs of an HC. The EAs of the harmonic of order i of $(.\overline{a_1}, .\overline{a_2})$ are given by

$$H^{(i)}(.\overline{a_1}, .\overline{a_2}) = \left(.\overline{a_1 \underbrace{a_2 a_2 \cdots a_2}_{i}}, .\overline{a_2 \underbrace{a_1 a_1 \cdots a_1}_{i}} \right). \tag{2.1}$$

When $i = 0, 1, 2, 3, \ldots$, (2.1) calculates a sequence of HCs and, when $i \to \infty$, becomes

$$H^{(\infty)}(.\overline{a_1}, .\overline{a_2}) = \left(.\overline{a_1 \underbrace{a_2 a_2 \cdots a_2}_{\infty}}, .\overline{a_2 \underbrace{a_1 a_1 \cdots a_1}_{\infty}} \right) = (.a_1\overline{a_2}, .a_2\overline{a_1}), \tag{2.2}$$

giving, as can be seen, preperiodic arguments. Therefore, they correspond to a Misiurewicz point.

Next, we are going to introduce pseudoharmonics, which are a generalization of harmonics.

2.1.2. Pseudoharmonics

Let $(.\overline{a_1}, .\overline{a_2})$ be the EAs of an HC, and let $(.\overline{b_1}, .\overline{b_2})$ be the EAs of another HC which is related to the first one, as will be seen later. The EAs of the pseudoharmonic of order i of $(.\overline{a_1}, .\overline{a_2})$ and $(.\overline{b_1}, .\overline{b_2})$ are

$$PH^{(i)}\left[(.\overline{a_1}, .\overline{a_2}); \left(.\overline{b_1}, .\overline{b_2}\right)\right] = \left(.\overline{a_1 \underbrace{b_2 b_2 \cdots b_2}_{i}}, .\overline{a_2 \underbrace{b_1 b_1 \cdots b_1}_{i}} \right). \tag{2.3}$$

When $i = 0, 1, 2, 3, \ldots$, (2.3) calculates again a sequence of HCs and, when $i \to \infty$, becomes

$$PH^{(\infty)}\left[(.\overline{a_1}, .\overline{a_2}); \left(.\overline{b_1}, .\overline{b_2}\right)\right] = \left(.\overline{a_1 \underbrace{b_2 b_2 \cdots b_2}_{\infty}}, .\overline{a_2 \underbrace{b_1 b_1 \cdots b_1}_{\infty}} \right) = \left(.a_1\overline{b_2}, .a_2\overline{b_1}\right). \tag{2.4}$$

Equation (2.4) calculates, as (2.2), a pair of EAs of a Misiurewicz point.

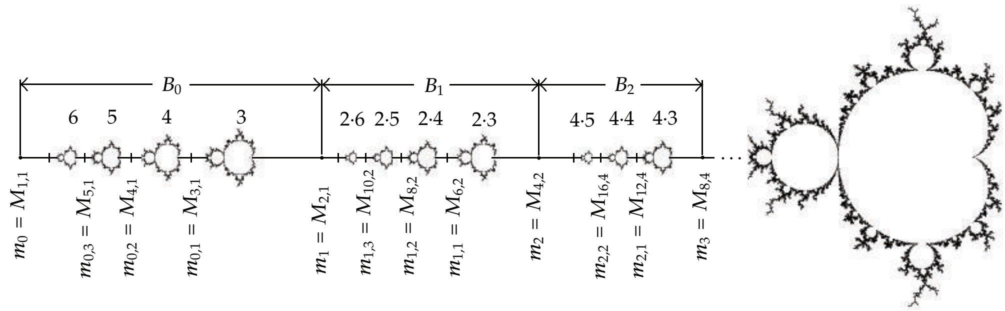


Figure 1: Mandelbrot set and a sketch of the antenna showing the first three chaotic bands B_0, B_1, and B_2 separated by the merging points m_i. In each chaotic band, the more important HCs and Misiurewicz points are shown.

When $(.\overline{b_1}, .\overline{b_2}) = (.\overline{a_1}, .\overline{a_2})$, (2.3) and (2.4) become (2.1) and (2.2), and then pseudoharmonics become harmonics. Therefore, harmonics are a particular case of pseudoharmonics. Pseudoharmonics are applied to two HCs while harmonics are applied to only one. However, according to what we have just seen above, harmonics can be considered as pseudoharmonics when the two HCs are the same HC.

2.2. Structure of the Mandelbrot Set

At this point we are going to show what we mean by structure of the Mandelbrot set. To do so, we base on a generalization of the structure of a one-dimensional quadratic map. Therefore, we will start seeing the structure of the one-dimensional case, and then we will generalize it to the Mandelbrot set with the help of shrubs already mentioned.

2.2.1. Structure of a One-Dimensional Quadratic Map

The concept of structure was introduced by us to study one-dimensional quadratic maps [24] for the case of HCs. Here we extend and systematize this concept, and we apply it to Misiurewicz points.

One-Dimensional Structural HCs

As we have just mentioned, the concept of structure is introduced by us for the first time to calculate the structure of a one-dimensional quadratic map [24]. Indeed, using our tool, the harmonics already introduced in Section 2.1.1, we calculate the symbolic sequences of the last appearance HCs, what we call LAHCs, of each chaotic band. Subsequently, we show that we could do the same with the EAs instead of with symbolic sequences [32, 33]. To study the one-dimensional quadratic map we used the real Mandelbrot map, which is the intersection of the Mandelbrot set with the x-axis, and to visualize it we used the Mandelbrot set antenna.

See Figure 1 which shows the Mandelbrot set and a sketch of the antenna showing the first three chaotic bands B_0, B_1, and B_2, which are separated by the known merging points of the chaotic bands, $m_1, m_2, m_3, \ldots$ (m_0 is the tip of the antenna) which we know are Misiurewicz points. In each of the chaotic bands four HCs are shown. In B_0, these HCs are last appearance HCs. The HCs of the other chaotic bands are not LAHCs in the strict sense but

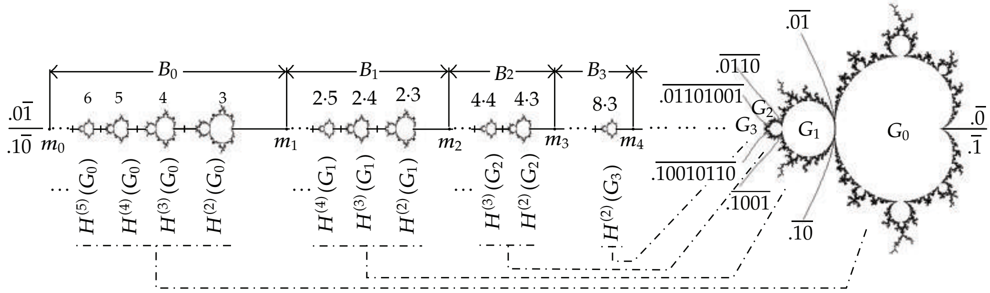


Figure 2: Mandelbrot set showing the main cardioid G_0 with the period doubling cascade G_1, G_2, $G_3, \ldots$ and a sketch of the antenna with the first four chaotic bands B_0, B_1, B_2, and B_3 separated by the merging points m_i. Each HC of B_i can be obtained by the successive harmonics of G_i.

they are LAHCs inside each of the chaotic bands, that is, they are local LAHCs, what we call LLAHCs. In each chaotic band B_i of Figure 1, some points $m_{i,j}$ between LLAHCs are shown. These points are Misiurewicz points that divide the chaotic bands in stretches.

Now let us focus our attention on Figure 2, which also shows the Mandelbrot set and a sketch of the antenna. In this figure we can see the main cardioid, G_0, and the discs of its period doubling cascade, $G_1, G_2, G_3, \ldots$, each one of them with its pair of EAs. Likewise, the chaotic bands B_0, B_1, B_2, and B_3 are depicted, each one of them with their first LLAHCs.

As we can see in [32] and as shown in Figure 2, the EAs of the LLAHCs of the chaotic band B_0 can be calculated as the successive harmonics of G_0. That is, G_0 can be considered the gene or generator of the chaotic band B_0. Likewise, the EAs of the LLAHCs of the chaotic band B_1 can be calculated as the successive harmonics of G_1; therefore this one can be considered the gene of the chaotic band B_1. Similarly, G_2 can be considered the gene of the chaotic band B_2, G_3 can be considered the gene of the chaotic band B_3, and so on. Hence, the chaotic bands $B_0, B_1, B_2, B_3, \ldots$ have their origins in the genes $G_0, G_1, G_2, G_3, \ldots$, because $H^{(i)}(G_0), H^{(i)}(G_1), H^{(i)}(G_2), H^{(i)}(G_3), \ldots$, where $2 \leq i \leq \infty$, calculate the external arguments of the local last appearance hyperbolic components of the chaotic bands $B_0, B_1, B_2, B_3, \ldots$.

These HCs just calculated in each chaotic band are the structural HCs of the corresponding chaotic band and determine, together with the structural Misiurewicz points that we will see later, the structure of such chaotic band.

As we know from Schleicher [34], an HC is narrow if it contains no component of equal or lesser period in its wake. In B_0, saying that an HC is an LAHC or LLAHC, is equivalent to saying that it is narrow. The LLAHCs of the other chaotic bands are not narrow in the strict sense, but they are narrow inside each chaotic band; that is, they are what we call locally narrow.

Structural HCs of a chaotic band are therefore characterized by the following.

(a) Their EAs can be calculated from the gene of the chaotic band by using harmonics.

(b) They are the local last appearance HCs (i.e., LLAHCs) of the chaotic band, which is equivalent to say that they are locally narrow.

(c) They are the lowest period (and largest size) HC in each stretch. That is why sometimes we call them the representative of the stretch.

How can we easily know whether an HC, which is identified by its pair of EAs, is structural or not? Or, what is the same, how can we easily know whether it is a LLAHC

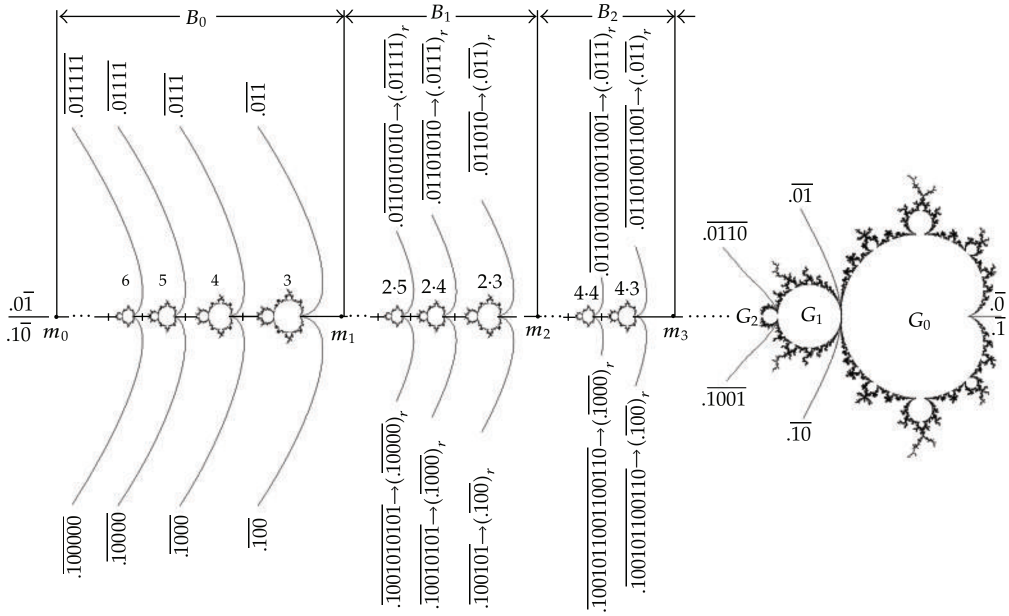


Figure 3: Mandelbrot set and a sketch of the antenna showing the first three chaotic bands B_0, B_1, and B_2 separated by the merging points m_i. In each chaotic band, the reduced EAs of the first LLAHCs are shown.

or locally narrow? To answer this question, let us see Figure 3, which shows again the Mandelbrot set and a sketch of the antenna with the chaotic bands $B_0, B_1, B_2, \ldots$. In each of these bands the first LLAHCs are given, each one of them with its pair of EAs. In the band B_0 we see that, for each of the LLAHCs, the two EAs are consecutive numbers (of course, they have the same number of digits). So, after the number $.\overline{011}$ it comes the number $.\overline{100}$, or after the number $.\overline{0111}$ it comes the number $.\overline{1000}$, with no possible intermediate value unless you increase the number of digits. Although this could be our criterion, this does not happen with the EAs of the other chaotic bands. Note, for example, the chaotic band B_1. In this band, the EAs of the LLAHC of period 6, $.\overline{011010}$ and $.\overline{100101}$, are not consecutive numbers.

To solve this problem, we introduce the *reduced EAs*. The gene or generator of B_1 is $G_1 = (.\overline{01}, .\overline{10})$ that we will take as a new unit. Therefore, in the LLAHCs of B_1, we put a 0 instead of 01, and we put a 1 instead of 10. Thus, the pair of reduced EAs of the LLAHC (with period 6) seen before is $r(.\overline{011010}, .\overline{100101}) = (.\overline{011}, .\overline{100})_r$. Then, although the EAs are not consecutive numbers, the reduced EAs are indeed consecutive numbers. Obviously, we can carry out a parallel process for any chaotic band as shown in the results of Figure 3. Therefore, the answer to our question is

An HC is structural if its reduced EAs are consecutive numbers.

Structural Misiurewicz Points

Now let us focus our attention on the Misiurewicz points of Figure 1 (although the binary expansions of the Misiurewicz points are not depicted in order to not overprint the figure). First, we can see the merging points m_i, which are the limits of the chaotic bands. In general, $m_i = H^{(\infty)}(G_i) = M_{2^i,2^{i-1}}$ for $i = 0, 1, 2, \ldots$ [27], a Misiurewicz point which is the upper limit of the chaotic band B_i (or merging point of the bands B_i and B_{i-1}). Let us note that the upper limit of the chaotic band B_0 is $m_0 = H^{(\infty)}(G_0) = M_{1,1/2}$. But as we know, if $m_0 = M_{1,1/2}$, also $m_0 = M_{1,1}$, a Misiurewicz point that is both the upper limit of B_0 and the tip of the antenna.

Second we can see $m_{i,j}$, those on which we focus now. Let us note that, in each chaotic band of the figure, we depict a Misiurewicz point between every two structural HCs. As mentioned before, these points divide the chaotic bands in stretches, that is, a stretch is the part of the band between two of these consecutive points. These Misiurewicz points can be calculated using the pseudoharmonics seen in Section 2.1.2.

Indeed, if we focus on the chaotic band B_0, the infinite pseudoharmonic of the period-3 structural HC with G_1 gives $m_{0,1}$ (as we will see in Section 2.3, G_1 is the second ancestor of the HCs of the chaotic band B_0). Since the EAs of the first one are $(.\overline{011}, .\overline{100})$ and the EAs of the second one are $(.\overline{01}, .\overline{10})$, we have $m_{0,1} = PH^{(\infty)}[(.\overline{011}, .\overline{100}); (.\overline{01}, .\overline{10})] = (.011\overline{10}, .100\overline{01})$, which is a Misiurewicz point $M_{3,1}$. Similarly, the infinite pseudoharmonic of the period-4 structural HC with G_1 gives $m_{0,2}$. Since the EAs of the first one are $(.\overline{0111}, .\overline{1000})$ and the EAs of the second one are $(.\overline{01}, .\overline{10})$, we have $m_{0,2} = PH^{(\infty)}[(.\overline{0111}, .\overline{1000}); (.\overline{01}, .\overline{10})] = (.0111\overline{10}, .1000\overline{01})$, which is a Misiurewicz point $M_{4,1}$ and so on for $m_{0,3}, m_{0,4}, \ldots$, which are Misiurewicz points $M_{5,1}, M_{6,1}, \ldots$.

If we focus now on the chaotic band B_1, the infinite pseudoharmonic of the period-6 structural HC with G_2 gives $m_{1,1}$. Since the EAs of the first and second ones are $(.\overline{011010}, .\overline{100101})$ and $(.\overline{0110}, .\overline{1001})$, respectively, we have $m_{0,1} = PH^{(\infty)}[(.\overline{011010}, .\overline{100101}); (.\overline{0110}, .\overline{1001})] = (.011010\overline{1001}, .100101\overline{0110})$, which is a Misiurewicz point $M_{6,2}$. Proceeding in the same way for $m_{1,2}, m_{1,3}, \ldots$, we obtain that they are Misiurewicz points $M_{8,2}, M_{10,2}, \ldots$. Similarly, in B_2 the $m_{2,1}$, $m_{2,2}$, $\ldots$, are Misiurewicz points $M_{12,4}, M_{16,4}, \ldots$, and, in general, in B_i the $m_{i,j}$ are Misiurewicz points $M_{(j+2)2^i,2^i}$.

Misiurewicz points we have just seen in each of the chaotic bands are the structural Misiurewicz points of these bands. Structural Misiurewicz points of a chaotic band are therefore characterized by the following.

(a) Their EAs can be calculated using the pseudoharmonics as we have shown above.

(b) All of them have the same periodic part, which coincides with the period of the generator of the band.

(c) The periodic part of any other nonstructural Misiurewicz point is greater than that of the structural Misiurewicz point.

(d) The preperiodic part of the upper end (the end farthest from the main cardioid) of a stretch coincides with the period of the structural HC of this stretch.

Taking into account what we have just seen, it is now elementary to answer the question of how easily to know if a Misiurewicz point is structural or not. The easiest criterion would be that, in the chaotic band B_i, a Misiurewicz point is structural if its periodic part is 2^i, that is the period of G_i, which is the generator of the chaotic band.

All the structural HCs and structural Misiurewicz points of a particular chaotic band constitute the structure of such a chaotic band. The structure of the real Mandelbrot set is that of all its chaotic bands.

Now let us move from the real case to the complex one; that is, our final aim is to know the structure of the Mandelbrot set, which is complex. Now also, the structure of the Mandelbrot set will be that of all its chaotic bands. However, as we will see, in this complex case the chaotic bands are not as simple as in the one-dimensional case and we need to use

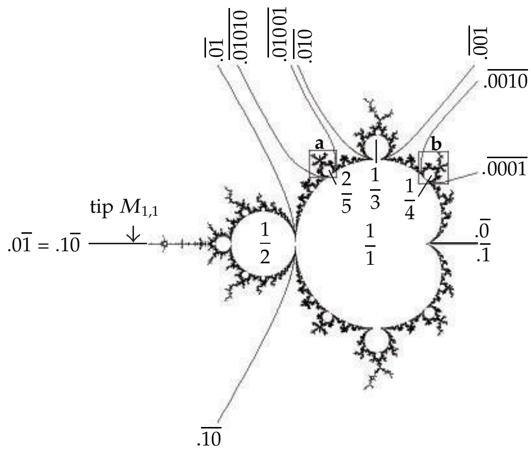


Figure 4: Mandelbrot set with some representative external arguments. Shrub (2/5) and shrub (1/4) are framed in rectangles **a** and **b**.

shrubs to solve the problem. Therefore, in the next section first we review what is a shrub in order to later see the structure of a shrub and then to see the structure of the Mandelbrot set.

2.2.2. Shrubs and Their Structure

Let us see what we mean by shrub, so called by the shaping of the chaotic region of the Mandelbrot set. In parallel we will see the structure of a shrub that somehow is going to be a generalization to the complex case of what we have seen for the real case. A general study of shrubs can be seen in [23].

Figure 4 shows the Mandelbrot set with the EAs of some HCs. The HCs which are directly in contact with the main cardioid are called primary HCs, and we represent them by q_1/p_1, for example, 1/2, 1/3, 1/4 and 2/5 of the figure. Likewise, the HCs which are directly in contact with the primary HCs are called secondary HCs, $(q_1/p_1) \cdot (q_2/p_2)$, the HCs which are directly in contact with the secondary HCs are called tertiary HCs, $(q_1/p_1) \cdot (q_2/p_2) \cdot (q_3/p_3)$, and so on, with $(q_1/p_1) \cdots (q_N/p_N)$ being an N-ary HC.

As known [23], a primary shrub is the shrub of a primary HC. Similarly, a secondary, tertiary,. . ., N-ary shrub is the shrub of a secondary, tertiary,. . ., N-ary HC. An N-ary shrub has N subshrubs. The subshrubs are equivalent to the chaotic bands, as we know from [35], and therefore each subshrub has a gene or generator.

Primary Shrubs

Let us start analyzing the primary shrubs, shrub (q_1/p_1). Since they are primary shrubs, they only have one subshrub. The generator of all the primary shrubs is the main cardioid [25].

Consider specifically the shrub (2/5) treated in [30, 31], which can be seen in Figure 5. Figure 5(a) shows a sketch of the shrub (2/5), and Figure 5(b) shows a magnification of the rectangle **a** in Figure 4 corresponding to shrub (2/5). Starting from the primary HC 2/5, following its period doubling cascade one reaches the Myrberg-Feigenbaum point from which one accesses to the chaotic region, which we call shrub because of its shape. A shrub is

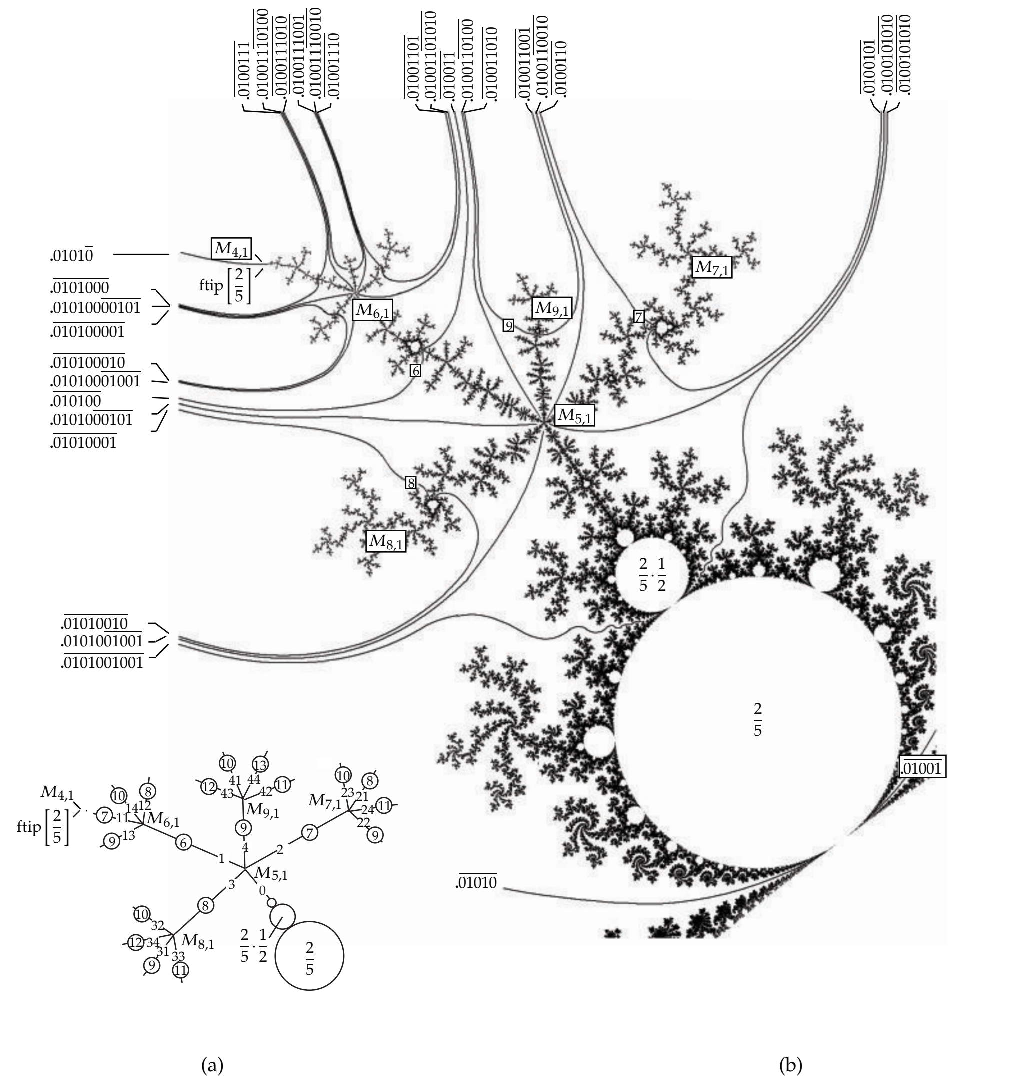


Figure 5: (a) Sketch of shrub (2/5). (b) Magnification of the rectangle **a** of Figure 4, corresponding to shrub (2/5).

extremely complex, so we represent only its structure, which is, as mentioned before, like the skeleton of the shrub.

As shown in the sketch of Figure 5(a), one starts from a branch 0, which reaches $M_{5,1}$, a branch point of 5 branches, the branch 0 of arrival and other four (1, 2, 3, and 4) of departure. This pattern is repeated indefinitely. Indeed, if we now consider, for example, the branch 2 as a branch of arrival, one reaches again a branch point of 5 branches, the arrival branch 2 and other four (21, 22, 23, and 24) departure branches. Then, in general, each branch point (all of them with p_1 branches) is reached by one branch of m digits and is exited through one of the other $p_1 - 1$ branches of $m + 1$ digits.

The branches of the complex plane are equivalent to the stretches on the real axis; that is, the branches are the generalization of the stretches. Likewise, branch points (or nodes) are the generalization to the complex plane of the stretches extremes of the real axis, $m_{i,j}$, which obviously are also branch points but only with two branches.

As already mentioned, the lowest period (and largest size) HC of each branch is called the representative of the branch. It is elementary to calculate its period, as we know from [23]. In the case of primary shrubs, all the branch representatives are narrow; that is, its two EAs are consecutive numbers.

As seen in the figure, all the Misiurewicz points of the nodes under consideration have period 1 (this is because the generator of this shrub is the main cardioid, 1/1, with period 1, as we know from [25]) and the preperiod is the same as the period of the representative of the arrival branch.

In these shrubs, we are considering two types of elements: the branch representatives and the nodes or branch points of the branch extremes. Representatives of branches are the structural HCs, as in the case of one-dimensional quadratic maps [24]. And the branch points are the structural nodes or structural Misiurewicz points. The set of structural nodes and structural HCs constitutes the structure of the primary shrub under consideration. To calculate the structure of the shrub is to calculate its structural HCs and its structural nodes.

In the shrub under consideration, each branch representative (or structural HC) has 2 EAs and each structural node has 5 EAs (p_1 EAs in general). When the EAs of the structural HCs are known, with the simple use of pseudoharmonics and pseudoantiharmonics all the EAs of structural nodes can be calculated directly, as discussed in [31]. But we insist that, in this paper, the EAs of the structural HCs of the branches are not known, and here we calculate the EAs of both the structural HCs of the branches and the structural nodes; that is, the entire structure will be calculated.

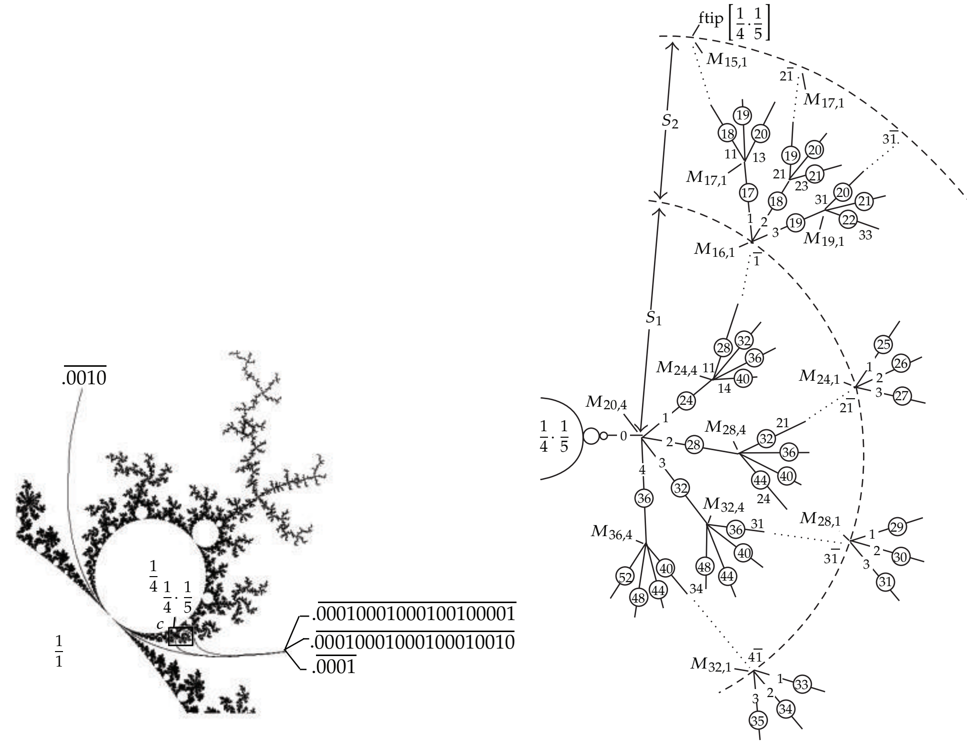
Let us point out some issues of nomenclature. First, in [23], and perhaps in some others, the preperiod of branch points was one unity more because we iterate from $z_0 = 0$ instead of from $z_0 = c$. Second, some authors call a primary HC a p/q whereas we call it q/p (so that p indicates period); that is, we interchange p and q. Third, an HC or a shrub should be written $(1/1) \cdot (q_1/p_1) \cdots (q_N/p_N)$ or shrub $((1/1) \cdot (q_1/p_1) \cdots (q_N/p_N))$; however, for our convenience often we write $(q_1/p_1) \cdots (q_N/p_N)$ or shrub $((q_1/p_1) \cdots (q_N/p_N))$.

Secondary Shrubs

Let us next consider secondary shrubs, shrub $((q_1/p_1) \cdot (q_2/p_2))$. Let us analyze, for example, $(1/4) \cdot (1/5)$, a secondary HC which is attached to 1/4, a primary HC which is marked by the rectangle **b** of Figure 4. The shrub of this HC, shrub $((1/4) \cdot (1/5))$, is shown in Figures 6 and 7. Figure 6(a), which is a magnification of the rectangle **b** of Figure 4, shows the primary disc 1/4 with some of its secondary discs, as $(1/4) \cdot (1/5)$ which is marked by the rectangle **c**, Figure 6(b) shows a sketch of the shrub $((1/4) \cdot (1/5))$, and Figure 7 shows a magnification of the rectangle **c** of Figure 6(a) corresponding to the shrub $((1/4) \cdot (1/5))$.

As we know [23], and as it is shown in the sketch of Figure 6(b), a secondary shrub has two subshrubs, S_1 and S_2. It is trivial to calculate the period of each branch representative or structural HC [23]. As can be seen in Figure 7, all the branch representatives of the subshrub S_2 are narrow; that is, their two EAs are consecutive numbers. We can not say the same for the case of S_1. However, the reduced EAs of the representatives are indeed consecutive numbers, which means that structural HCs are locally narrow.

As also seen in Figure 7, all the Misiurewicz points of structural nodes have period 4 in S_1 and period 1 in S_2 (it is because the generator of S_2 is the main cardioid, of period 1, and the generator of S_1 is 1/4, of period 4 [25]). Likewise, the preperiod of these Misiurewicz



(a) (b)

Figure 6: (a) Magnification of the rectangle **b** of Figure 4, corresponding to shrub (1/4). Shrub ((1/4) · (1/5)) is framed in the rectangle **c**. (b) Sketch of shrub ((1/4) · (1/5)).

points is the same as the period of the structural HC of the arrival branch of the node. In S_1 the structural nodes are branch points of 5 branches (p_2 branches in general), while in S_2 the structural nodes are branch points of 4 branches (p_1 branches in general).

N-Ary Shrubs

Generalizing, every N-ary shrub, shrub $((1/1) \cdot (q_1/p_1) \cdots (q_N/p_N))$, has N subshrubs $S_1 \cdots S_N$. Each structural node of a given subshrub S_i is a branch point of p_{N-i+1} branches and therefore has p_{N-i+1} EAs. The period of each structural HC of a given subshrub S_i is easily calculated [23]. The structural HCs are narrow in the last subshrub and locally narrow in all the others. As to the structural nodes, they have the same preperiod as the period of the structural HCs of the arrival branch to the node, and they have the same period as the period of the generator of S_i, that is, the period of $(1/1) \cdot (q_1/p_1) \cdots (q_{N-i}/p_{N-i})$, which is $1 \cdot p_1 \cdots p_{N-i}$.

Therefore, to see the structure of a shrub we have to see the structure of all the subshrubs, the structure of each subshrub being the set of all its structural HCs and structural Misiurewicz points.

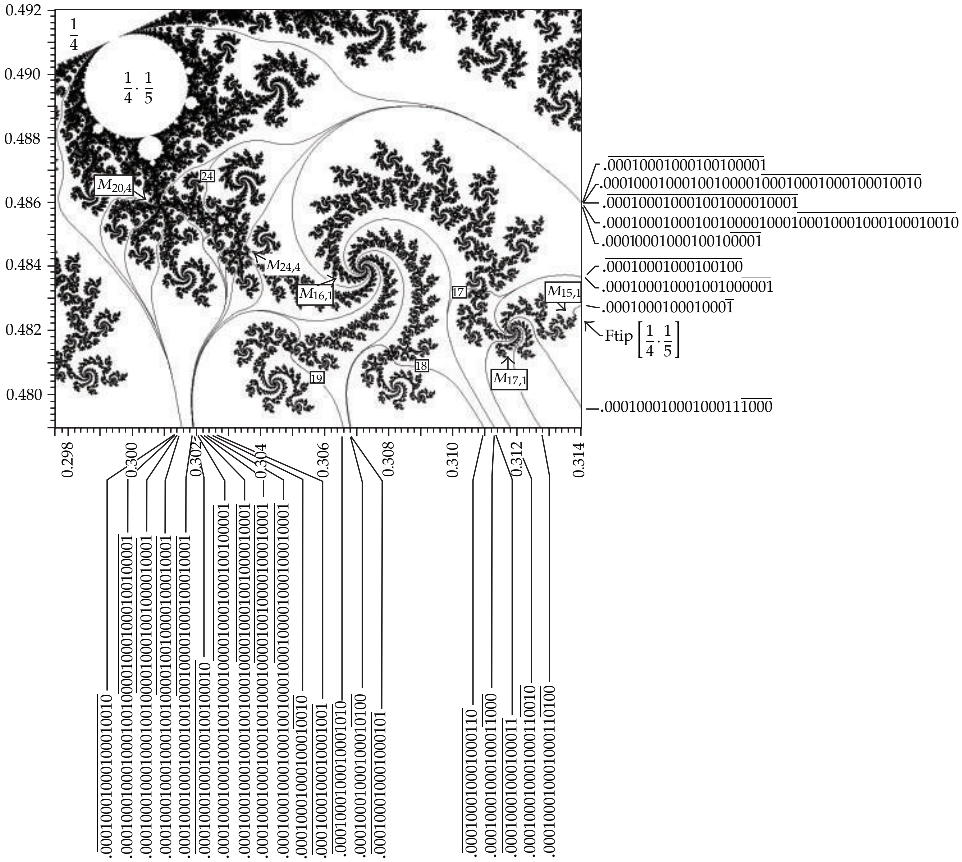


Figure 7: Magnification of the rectangle **c** of Figure 6(a), corresponding to shrub $((1/4) \cdot (1/5))$.

Structural HCs of a subshrub are characterized by the following.

(a) The period of the structural HC of each branch is easily calculated (not its EAs that can only be calculated directly, using pseudoharmonics, in the case of some branches). One of the objectives of this paper is to give an algorithm for its calculation.

(b) They are narrow HCs within each subshrub.

(c) They are the lowest period (and largest size) HC in each branch, so it is sometimes called the representative of the branch.

Similarly, structural Misiurewicz points of a subshrub S_i of an N-ary shrub are characterized by the following.

(a) They have p_{N-i+1} EAs. The algorithm for calculating the EAs will be given in this paper.

(b) They have the same periodic part, which coincides with the period of the generator of the subshrub.

(c) The period of any nonstructural Misiurewicz point is greater than the period of the structural Misiurewicz points.

(d) The preperiod of the upper extreme (the farthest extreme from the main cardioid) of a branch coincides with the period of the structural HC of this branch.

The criteria to easily know if an HC or a Misiurewicz point is structural are similar to those we saw in the one-dimensional case, namely,

(i) an HC is structural if its reduced EAs are consecutive numbers,

(ii) a Misiurewicz point belonging to the subshrub S_i is structural if it has the same period as the period of the generator of S_i.

2.3. *Ancestral Route*

For calculating pseudoharmonics, as known [31, 32], $(.\overline{b_1}, .\overline{b_2})$ has to be related to $(.\overline{a_1}, .\overline{a_2})$. Indeed, $(.\overline{b_1}, .\overline{b_2})$ has to be an HC of the ancestral route of $(.\overline{a_1}, .\overline{a_2})$. An N-ary shrub, shrub $((1/1)\cdot(q_1/p_1)\cdot(q_2/p_2)\cdots(q_N/p_N))$, has N subshrubs $S_i((1/1)\cdot(q_1/p_1)\cdot(q_2/p_2)\cdots(q_N/p_N))$, $1 \leq i \leq N$, and the generator of $S_i((1/1)\cdot(q_1/p_1)\cdot(q_2/p_2)\cdots(q_N/p_N))$ is $(1/1)\cdot(q_1/p_1)\cdot(q_2/p_2)\cdots(q_{N-i}/p_{N-i})$. The ancestral route of an HC is the ordered sequence of all its ancestors [31, 33, 35]. In this paper we are going to use only the first and second ancestors of the ancestral route. The first ancestor is the generator of S_i: $(1/1)\cdot(q_1/p_1)\cdots(q_{N-i}/p_{N-i})$; therefore, the second ancestor is $(1/1)\cdot(q_1/p_1)\cdots(q_{N-i+1}/p_{N-i+1})$.

2.3.1. $(.\overline{b_1}, .\overline{b_2})$ *Is the First Ancestor of* $(.\overline{a_1}, .\overline{a_2})$

Let $(.\overline{a_1}, .\overline{a_2})$ be the representative of the branch $d_1d_2\cdots d_m$ [23] of S_i. Let $(.\overline{b_1}, .\overline{b_2})$ be the first ancestor of $(.\overline{a_1}, .\overline{a_2})$, which is the generator $(1/1)\cdot(q_1/p_1)\cdots(q_{N-i}/p_{N-i})$ of S_i. In this case $PH^{(\infty)}[(.\overline{a_1}, .\overline{a_2}); (.\overline{b_1}, .\overline{b_2})] = (.a_1\overline{b_2}, .a_2\overline{b_1})$ calculates the first and last EAs of the Misiurewicz point placed in the upper limit of the branches $d_1d_2\cdots d_m11\cdots$ [36]. Therefore, $PH^{(\infty)}[(.\overline{a_1}, .\overline{a_2}); (.\overline{b_1}, .\overline{b_2})] = (.a_1\overline{b_2}, .a_2\overline{b_1})$ calculates an upper limit of S_i.

Examples

(a) For a primary shrub, as shrub $(2/5)$, $N = 1$ and $i = 1$, the first ancestor is $1/1$ whose EAs are $(.\overline{b_1}, .\overline{b_2}) = (.\overline{0}, .\overline{1})$. The EAs of $2/5$ are $(.\overline{a_1}, .\overline{a_2}) = (.\overline{01001}, .\overline{01010})$. Hence, $PH^{(\infty)}[(.\overline{01001}, .\overline{01010}); (.\overline{0}, .\overline{1})] = (.0100\overline{1}, .0101\overline{0})$, which are two EAs with the same value, which correspond to a $M_{4,1}$ Misiurewicz point, the ftip$(2/5)$ [23] as shown in Figure 5.

(b) For a secondary shrub, as shrub $((1/4)\cdot(1/5))$ of Figures 6 and 7, $N = 2$.

(b1) If we are in the first subshrub S_1, $i = 1$ and the first ancestor is $1/1\cdot q_1/p_1$, here $1/4$, whose EAs are $(.\overline{b_1}, .\overline{b_2}) = (.\overline{0001}, .\overline{0010})$. If we start, for example, from the period-24 representative of the branch 1, then $(.\overline{a_1}, .\overline{a_2}) = (.\overline{000100010001000100100010}, .\overline{000100010001001000010001})$. Hence, $PH^{(\infty)}[(.\overline{000100010001000100100010}, .\overline{000100010001001000010001}); (.\overline{0001}, .\overline{0010})] = (.0001000100010001\overline{0010}, .0001000100010010\overline{0001})$, which correspond to the first and last EAs of an $M_{16,1}$ Misiurewicz point, which is one of the upper limits of S_1, $\overline{1}$, where a portion of S_2 emerges, as shown in Figures 6 and 7.

(b2) If we are in the second subshrub, S_2, $i = 2$ and the first ancestor is 1/1, whose EAs are $(.\overline{b_1}, .\overline{b_2}) = (.\overline{0}, .\overline{1})$. If we start, for example, from the period-17 representative of the branch 1 of the portion that comes from $\overline{1}$, then $(.\overline{a_1}, .\overline{a_2}) = (.\overline{00010001000100011}, .\overline{00010001000100100})$. Hence, $PH^{(\infty)}[(.\overline{00010001000100011}, .\overline{00010001000100100}); (.\overline{0}, .\overline{1})] = (.0001000100010001\overline{1}, .0001000100010010\overline{0})$, two EAs with the same value, which correspond to a $M_{15,1}$ Misiurewicz point, the ftip$((1/4) \cdot (1/5))$ shown in Figures 6 and 7.

For a tertiary, quaternary,. . ., N-ary shrub the procedure will be the same.

2.3.2. $(.\overline{b_1}, .\overline{b_2})$ *Is the Second Ancestor of* $(.\overline{a_1}, .\overline{a_2})$

Let $(.\overline{a_1}, .\overline{a_2})$ be again the representative of the branch $d_1 d_2 \cdots d_m$ of S_i, and let $(.\overline{b_1}, .\overline{b_2})$ be now the second ancestor of $(.\overline{a_1}, .\overline{a_2})$, $(1/1) \cdot (q_1/p_1) \cdots (q_{N-i+1}/p_{N-i+1})$. In this case, $PH^{(\infty)}[(.\overline{a_1}, .\overline{a_2}); (.\overline{b_1}, .\overline{b_2})] = (.a_1\overline{b_2}, .a_2\overline{b_1})$ calculates the first and last EAs of the Misiurewicz point placed in the upper end of the branch $d_1 d_2 \cdots d_m$ [31, 33, 35].

Examples

(a) For a primary shrub, as shrub(2/5) of Figure 5, $N = 1$ and $i = 1$, the second ancestor is $(1/1) \cdot (q_1/p_1)$, here 2/5, whose EAs are $(.\overline{b_1}, .\overline{b_2}) = (.\overline{01001}, .\overline{01010})$. If we want to calculate the first and last EAs of the Misiurewicz point placed in the upper end of the representative of the branch 0, both the second ancestor and the representative are the same, $(.\overline{b_1}, .\overline{b_2}) = (.\overline{a_1}, .\overline{a_2}) = (.\overline{01001}, .\overline{01010})$. Hence, $PH^{(\infty)}[(.\overline{01001}, .\overline{01010}); (.\overline{01001}, .\overline{01010})] = (.01001\overline{01010}, .01010\overline{01001})$ are the first and last EAs of a $M_{5,1}$ Misiurewicz point, which is the upper end of the branch 0.

(b) For a secondary shrub, as shrub $((1/4) \cdot (1/5))$ of Figures 6 and 7, $N = 2$.

(b1) If we are in the first subshrub, $i = 1$ and the second ancestor is $(1/1) \cdot (q_1/p_1) \cdot (q_2/p_2)$, here $(1/4) \cdot (1/5)$, whose EAs are $(.\overline{b_1}, .\overline{b_2}) = (.\overline{00010001000100010010}, .\overline{00010001000100100001})$. If we start from the representative of the branch 0 of S_1, again both the second ancestor and the representative are the same, $(.\overline{a_1}, .\overline{a_2}) = (.\overline{b_1}, .\overline{b_2})$. Hence, $PH^{(\infty)}[(.\overline{00010001000100010010}, .\overline{00010001000100100001}); (.\overline{00010001000100010010}, .\overline{00010001000100100001})] = (.00010001000100010010\overline{00010001000100100001}, .00010001000100100001\overline{00010001000100010010})$, which correspond to the first and last EAs of a $M_{20,4}$ Misiurewicz point, which is the upper end of the branch 0 of S_1 (see Figures 6 and 7).

(b2) If we are in the second subshrub, $i = 2$ and the second ancestor is $(1/1) \cdot (q_1/p_1)$, here 1/4, whose EAs are $(.\overline{b_1}, .\overline{b_2}) = (.\overline{0001}, .\overline{0010})$ (see Figure 6(a)). If we start from the period-17 representative of the branch 1 of the portion that comes from $\overline{1}$, then $(.\overline{a_1}, .\overline{a_2}) = (.\overline{00010001000100011}, .\overline{00010001000100100})$. Hence, $PH^{(\infty)}[(.\overline{00010001000100011}, .\overline{00010001000100100}); (.\overline{01001}, .\overline{01010})] = (.00010001000100011\overline{01010}, .00010001000100100\overline{01001})$, which correspond to the first and last EAs of a $M_{17,1}$ Misiurewicz point, which is the upper end of the branch 1 of the portion of S_2 that comes from $\overline{1}$ (see Figures 6 and 7).

We will proceed in the same way for a tertiary,..., N-ary shrub.

3. Calculation of the EAs of the Structure of a Shrub

The aim of this paper is the calculation of the EAs of the structure of any shrub because in this case we can calculate the structure of all the Mandelbrot set. To do this, as mentioned in the introduction, we give a three-step algorithm to calculate the structure of a shrub in the Mandelbrot set. The first step uses pseudoharmonics, already introduced by us in [31] and reviewed in Section 2.1.2. The second and third steps need new contributions that will be introduced in the next two Sections 3.1 and 3.2, respectively. Finally, Section 3.3 introduces the three-step algorithm.

3.1. Calculation of the Intermediate External Arguments

Once you know the first and last EAs of a node, Devaney and Moreno-Rocha [30] developed a method for calculating intermediate EAs, the DM-R method. They applied this method only to the main node of a primary shrub although it can also be applied to any node of such a primary shrub. However, the method can not be applied to secondary, tertiary,... shrubs. We will show here a method for calculating intermediate EAs, which is valid not only for a primary shrub but also for the case of secondary, tertiary, and in general N-ary shrubs. To do this, let us see next two general properties of EAs of a Misiurewicz point. Based on the first property, we can calculate the periodic part, and, based on the second property, we can calculate the preperiodic part.

Let us consider any structural node from which b structural branches emerge. This node is a Misiurewicz point $M_{n,p}$, where n and p are, respectively, the preperiod and the period. As we know, this node has b preperiodics EAs, each one with a preperiod of n digits and a period of p—or multiple of p—digits. Let us see the first property corresponding to periods.

Property 1. If the b EAs of a node are arranged from the lowest to the highest, then their periods perform a left cyclic shift by β positions.

Indeed, let us see for example the main node of the shrub $(2/5)$ of Figure 5, and let us arrange its 5 EAs from the lowest to the highest: $.01001\overline{01010}$, $.01001\overline{10010}$, $.01001\overline{10100}$, $.01010\overline{00101}$, and $.01010\overline{01001}$. The periods of these 5 EAs are $\overline{01010}$, $\overline{10010}$, $\overline{10100}$, $\overline{00101}$, and $\overline{01001}$, respectively. Each of these periods can be obtained from the preceding one (and the first one from the last one) by moving the first 3 digits from the beginning to the end. Therefore, here $\beta = 3$.

Remark 3.1. If two consecutive EAs of the cycle are known, then β can be calculated and therefore the periodic parts of the other EAs can also be calculated.

This is what happens when the first and last (or two consecutive) EAs are known, because we can calculate β if we analyze how the last EA is turned into the first EA (or any EA into the next one). In the case we have just seen, the periodic part of the last external argument is $\overline{01001}$ and that of the first one is $\overline{01010}$. Since in order to turn $\overline{01001}$ into $\overline{01010}$ we had to move the first three digits to the end, we have $\beta = 3$, as seen before. This result can be used to calculate the periodic part of the second, third, and fourth EAs. β could also be calculated by the DM-R method by using the sums of Farey, but the DM-R method can only be applied to primary shrubs, whereas our method can be applied in general to N-ary shrubs.

Being fully general, this method can also be applied to secondary, tertiary,... shrubs. Let us apply it, for example, to the branch point of the branches 0, 1, 2, 3, and 4 of the subshrub S_1 of the shrub $((1/4) \cdot (1/5))$, which is a Misiurewicz point $M_{20,4}$, where 5 EAs land (see Figures 6 and 7). We assume the last and first EAs are known. To turn the periodic part of the last external argument , $\overline{00010001000100010010}$, into the periodic part of the first EA, $\overline{00010001000100100001}$, we have to move the first four digits to the end. So $\beta = 4$. Using this result, we calculate the periodic part of the second, third, and fourth EAs, which are $\overline{00010001001000010001}$, $\overline{00010010000100010001}$, and $\overline{00100001000100010001}$. It is of no interest to apply it to more cases because the procedure is always the same. Therefore, let us see now the second property concerning the preperiods.

Property 2. The preperiods of the b EAs of a node arranged from the lowest one to the highest one have only two possible values: the preperiod of the first EA, which is kept while the periodic part is increasing, and another second preperiod that appears and remains until the last EA, when the periodic part decreases.

Examples

(a) Let us first apply Property 2 to a primary shrub. For example, let us continue with the main node of the shrub $(2/5)$ in Figure 5. As just calculated, the periods for the 5 EAs (these EAs arranged from the lowest to highest) are $\overline{01010}$, $\overline{10010}$, $\overline{10100}$, $\overline{00101}$, and $\overline{01001}$, respectively. The preperiods of the first one and last one are known: .01001 and .01010. The preperiods of the intermediate ones have to be one of these two, being one or the other according to Property 2. As can be seen in the previous arrangement, going from the first period to the second period the period value increases; then the preperiod of the second EA is the same as that of the first EA. Similarly, going from the second EA to the third EA the value of the period increases; then the preperiod of the third EA is the same as that of the first EA. On the contrary, going from the third EA to the fourth EA the value of the period decreases; then the preperiod of the fourth EA becomes the same as the preperiod of the last EA. Therefore, the full results arranged from the first to last are $.01001\overline{01010}$, $.01001\overline{10010}$, $.01001\overline{10100}$, $.01010\overline{00101}$, and $.01010\overline{01001}$.

(b) Let us now apply Property 2 to a secondary shrub. For example, let us continue with the branch point of the branches 0, 1, 2, 3, and 4 of the subshrub S_1 of the shrub $((1/4) \cdot (1/5))$, which is a Misiurewicz point $M_{20,4}$, where 5 EAs land (see Figures 6 and 7). As just calculated, the periods of the 5 EAs arranged from the lowest to highest are $\overline{00010001000100100001}$, $\overline{00010001001000010001}$, $\overline{00010010000100010001}$, $\overline{00100001000100010001}$, and $\overline{00010001000100010010}$. The preperiods of the first EA and last EA are known: .0001000100010001001 and .00010001000100100001, respectively. The preperiods of the intermediate ones have to be one of these two. As can be seen in the previous arrangement, going from first to second, from second to third, and from third to fourth, the period value increases; then the preperiods of the second, third, and fourth EAs are the same as that of the first EA. Then the full results arranged from the first to last are $.0001000100010001001\overline{000010001000100100001}$, $.0001000100010001001\overline{000010001001000010001}$, $.0001000100010001001\overline{000010010000100010001}$, $.0001000100010001001\overline{000100001000100010001}$, and $.00010001000100100001\overline{00010001000100010010}$.

It is not worth setting an example of a tertiary,. . ., N-ary shrub because the procedure is always the same.

3.2. Calculation of the External Arguments of a Structural Hyperbolic Component

Let us see a new property which will allow us to calculate the EAs of the representatives, or structural HCs, of a branch.

Let $(.\overline{a}_1, .\overline{a}_2)$ be the periodic EAs of the period-p representative of a branch. Let $(.b_1\overline{c}_1, .b_2\overline{c}_2)$ be the two preperiodic EAs of the lower extreme of the branch which are closest to $(.\overline{a}_1, .\overline{a}_2)$, and let $(.d_1\overline{e}_1, .d_2\overline{e}_2)$ be the two preperiodic EAs of the upper extreme of the branch which are closest to $(.\overline{a}_1, .\overline{a}_2)$ (the lower/upper extreme is the nearest/farthest from the main cardioid).

Property 3. In each of the two groups $\{.\overline{a}_1, .b_1\overline{c}_1, .d_1\overline{e}_1\}$ and $\{.\overline{a}_2, .b_2\overline{c}_2, .d_2\overline{e}_2\}$ the three EAs have the same first p digits.

Remark 3.2. If we know $.b_1\overline{c}_1$ or $.d_1\overline{e}_1$, then we also know $.\overline{a}_1$. Likewise, if we know $.b_2\overline{c}_2$ or $.d_2\overline{e}_2$, then we also know $.\overline{a}_2$.

Examples

(a) Let us see an example of a primary shrub, as usual the shrub (2/5) of Figure 5. Look at the branch 1, whose branch representative has period 6 and whose upper and lower extremes are the Misiurewicz points $M_{5,1}$ and $M_{6,1}$. The EAs of the representative of the branch are $(.\overline{010011}, .\overline{010100})$, obviously both periodic ones, the first one being on the right of the branch and the second one on the left of the branch. From the lower extreme of the branch 5 EAs emanate, in this case preperiodic ones, but we will consider only two of them, those nearest to the EAs of the branch representative, which in this case are the third and fourth EAs of $M_{5,1}$. Likewise, from the upper extreme of the branch 5 EAs emanate, again preperiodic ones, but we will consider only two of them, those closest to the EAs of the branch representative, which in this case are the first and last EAs of $M_{6,1}$. In total we have six EAs, and we will group the three on the right of the branch (right group) and the three on the left of the branch (left group). The right and left groups are $\{.\overline{010011}; .01001\overline{10100}; .01001\overline{101010}\}$ and $\{.\overline{010100}; .0101\overline{000101}; .01010\overline{001001}\}$, respectively. As we can see, in both the right one and the left one, the first 6 digits of the 3 EAs in each group match, as indicated by Property 3. Therefore, in any of the previous two groups, if one knows the preperiodic EA from one of the two extremes of the branch one also knows the corresponding periodic EA of the branch representative, as indicated by Remark 3.2.

(b) Let us see next an example of a branch of a secondary shrub, namely, the branch 1 of the subshrub S_1 of the shrub $((1/4) \cdot (1/5))$ of Figures 6 and 7 with which we are working. The representative of the branch 1 has period 24 and the lower and upper extremes are Misiurewicz points $M_{20,4}$ and $M_{24,4}$. As we can see in Figure 7, the right and left groups are $\{.\overline{000100010001000100100010}$; $.00010001000100010010\overline{001000010001000100010001}$; $.000100010001000100100010\overline{00010001000100100001}\}$ and $\{.\overline{000100010001001000010001}$; $.00010001000100100001\overline{00010001000100010010}$; $.000100010001001000010001\overline{00010001000100010010}\}$. As we can see again, in both the right group and the left group the first 24 digits of the three EAs in each group match. Therefore, in any of the previous two groups, if one knows the preperiodic EA of one of the two extremes of the branch, one also knows the corresponding periodic EA of the branch representative.

Again, it is not worth analysing the example of a tertiary,..., N-ary shrub because the procedure is always the same.

3.3. *Three-Step Algorithm to Calculate the External Arguments of the Structure of a Shrub*

Using what we have seen so far, both in Section 2 (where small new contributions are introduced) and in Sections 3.1 and 3.2 (which contain totally new contributions), we can calculate all the EAs of the structure of any shrub $((q_1/p_1)\cdots(q_N/p_N))$, which is the aim of this paper.

To show the calculation algorithm in a general way, let us suppose that we start from the representative of the branch $d_1d_2\cdots d_m$ of the subshrub S_i, whose EAs, $(.\overline{a_1}, .\overline{a_2})$, are known. The calculation algorithm consists of three steps given below.

(1) Using pseudoharmonics reviewed in Section 2.1.2, we calculate either the first and last EAs of the upper extreme of the branch (when $(.\overline{b_1}, .\overline{b_2})$ is the second ancestor of $(.\overline{a_1}, .\overline{a_2})$) or the first and last EAs of the upper extreme of $d_1d_2\cdots d_m 11\cdots$, which is either a tip or a point where a portion of a new subshrub is born (when $(.\overline{b_1}, .\overline{b_2})$ is the first ancestor of $(.\overline{a_1}, .\overline{a_2})$).

(2) Using Properties 1 and 2 introduced in Section 3.1, we calculate the EAs which are intermediate between the first one and the last one calculated in step (2.1).

(3) Using Property 3 of Section 3.2, we calculate the EAs $(.\overline{a_1'}, .\overline{a_2'})$ of the representative of a branch $d_1d_2\cdots d_m d_{m+1}$ with one digit more than the preceding one (of course, the lower extreme of the branch $d_1d_2\cdots d_m d_{m+1}$ is the upper extreme of the branch $d_1d_2\cdots d_m$).

Obviously, if we want to calculate all the EAs of the structure of any shrub $((q_1/p_1)\cdots(q_N/p_N))$, which was the aim of this paper, we should start from the branch 0 instead of starting from a general branch $d_1d_2\cdots d_m$. But the representative of the branch 0 is $(q_1/p_1)\cdots(q_N/p_N)$ whose EAs are always known, as we will see in the following examples.

Examples

(a) Example of a Primary Shrub

Let us consider a primary shrub, shrub (q_1/p_1); for example let us see again the shrub (2/5) of Figure 5. Suppose that we do not know any data of the structure of the shrub with the exception of the value of the EAs of the starting HC, $(.\overline{a_1}, .\overline{a_2}) = (.\overline{01001}, .\overline{01010})$, which is the representative of the branch 0. Note that this HC is a disc of the periodic part of the Mandelbrot set, and the EAs of these discs are always known because they can be calculated by either the Schleicher algorithm [37], if it is a primary disk, or with both the Schleicher algorithm and the tuning algorithm [38], if it is a secondary disk or a higher order one. As seen in Section 2.3, the infinite pseudoharmonic of $(.\overline{a_1}, .\overline{a_2}) = (.\overline{01001}, .\overline{01010})$ and its second ancestor, which is itself, calculate the first and last EAs of the upper extreme of the branch 0, obtaining $PH^{(\infty)}[(.\overline{01001}, .\overline{01010}); (.\overline{01001}, .\overline{01010})] = (.01001\overline{01010}, .01010\overline{01001})$, which are the first and last EAs of the 5 EAs of the Misiurewicz point $M_{5,1}$.

As seen in Section 3.1, using Properties 1 and 2 we calculate the intermediate EAs of such an upper extreme of the branch 0, and we obtain that the 5 EAs of that point, arranged from the lowest to highest, are $.01001\overline{01010}$, $.01001\overline{10010}$, $.01001\overline{10100}$, $.01010\overline{00101}$, and $.01010\overline{01001}$. As seen in Section 3.2, applying Property 3 we can calculate the EAs of the representatives of branches 1, 2, 3, and 4. Indeed, taking into account the third and fourth EAs, we calculate the period-6 representative of the branch 1, $(.\overline{a'_1}, .\overline{a'_2})_{b1} = (.\overline{010011}, .\overline{010100})$; taking into account the first and second EAs, we calculate the period-7 representative of the branch 2, $(.\overline{a'_1}, .\overline{a'_2})_{b2} = (.\overline{0100101}, .\overline{0100110})$; taking into account the fourth and fifth EAs, we calculate the period-8 representative of the branch 3, $(.\overline{a'_1}, .\overline{a'_2})_{b3} = (.\overline{01010001}, .\overline{01010010})$; taking into account the second and third EAs, we calculate the period-9 representative of the branch 4 $(.\overline{a'_1}, .\overline{a'_2})_{b4} = (.\overline{010011001}, .\overline{010011010})$.

From here, we can repeat the process indefinitely in order to calculate the whole structure of the shrub (2/5). Indeed, by considering for example the branch 1, the first step calculates the first and last EAs of the upper extreme of the branch, the second step calculates the intermediate EAs, and the third step calculates the EAs of the representatives of branches 11, 12, 13, and 14. If we start from the branch 2, the first two steps calculate the EAs of the upper extreme of the branch and the third step calculates the EAs of the representatives of branches 21, 22, 23, and 24. We operate in the same way if we start from the branches 3 or 4 to finally calculate the representatives of branches 31, 32, 33, and 34, or branches 41, 42, 43, and 44. Starting now from any of the branches of two digits, for example, from the branch 32, with the first two steps the EAs of the upper extreme of the branch are calculated and with the third step the EAs of the representatives of branches 321, 322, 323, and 324 are calculated. And so on.

(b) Example of a Secondary Shrub

Let us consider a secondary shrub, shrub $((q_1/p_1) \cdot (q_2/p_2))$, for example the repeatedly used shrub $((1/4) \cdot (1/5))$ which can be seen in Figures 6 and 7. The EAs of the secondary HC $(1/4) \cdot (1/5)$ (it is a disk of the periodic region, therefore its EAs can be calculated by using the Schleicher algorithm and the tuning algorithm) are $(.\overline{a_1}, .\overline{a_2}) = (.\overline{00010001000100010010}, .\overline{00010001000100100001})$. Since it is a secondary shrub, $N = 2$, and therefore it has two subshrubs, S_1 and S_2 that we calculate below.

(b1) First Subshrub, S_1

Let us first calculate the structure of the first subshrub. First of all, we have to calculate the first and last EAs of the upper extreme of the branch 0. Therefore, the pair $(.\overline{b_1}, .\overline{b_2})$ has to be the EAs of the second ancestor of $(.\overline{a_1}, .\overline{a_2})$, that is, $(1/1) \cdot (q_1/p_1) \cdots (q_{N-i+1}/p_{N-i+1})$ where $N = 2$ and $i = 1$; or, what is the same, the EAs of $(1/4) \cdot (1/5)$. Then $(.\overline{b_1}, .\overline{b_2}) = (.\overline{a_1}, .\overline{a_2})$. Therefore, as we know from Sections 2.1.2 and 2.3, $PH^{(\infty)}[(.\overline{00010001000100010010}, .\overline{00010001000100100001});(.\overline{00010001000100010010}, .\overline{00010001000100100001})] = (.00010001000100010010\overline{00010001000100100001}, .00010001000100100001\overline{00010001000100010010})$, which correspond to the first and last EAs of a $M_{20,4}$ Misiurewicz point, which is the upper extreme of the branch 0 of S_1. As discussed in Section 3.1, using Properties 1 and 2 we calculate the intermediate EAs of this upper extreme of the branch 0 and we obtain that the 5 EAs, arranged from the lowest to highest, are $.00010001000100010010\overline{00010001000100100001}$, $.00010001000100010010\overline{00010001001000010001}$, $.00010001000100010010\overline{00010010000100010001}$, $.00010001000100010010\overline{00100001000100010001}$, and $.00010001000100100001\overline{00010001000100010010}$. As discussed in Section 3.2, by applying Property 3 we can calculate the EAs of the representatives of the branches 1, 2, 3, and 4, whose periods are 24, 28, 32, and 36,respectively. Indeed, by taking into account the fourth and fifth EAs, we calculate the representative of the period-24 branch 1, $(.\overline{a'_1}, .\overline{a'_2})_{b1} = (.\overline{000100010001000100100010}, .\overline{000100010001001000010001})$; by taking into account the third and fourth EAs, we calculated the representative of the period-28 branch 2, $(.\overline{a'_1}, .\overline{a'_2})_{b2} = (.\overline{0001000100010001001000010010}, .\overline{0001000100010001001000100001})$; by taking into account the second and third EAs, we calculate the period-32 representative of the branch 3, $(.\overline{a'_1}, .\overline{a'_2})_{b3} = (.\overline{00010001000100010010000100010010}, .\overline{00010001000100010010000100100001})$; by taking into account the first and second EAs, we calculate the period-36 representative of the branch 4, $(.\overline{a'_1}, .\overline{a'_2})_{b4} = (.\overline{000100010001000100100001000100010010}, .\overline{000100010001000100100001000100100001})$ (for reasons of space, Figure 7 only shows the EAs of the period-24 representative).

From here, we can repeat the process indefinitely in order to calculate the whole structure of the first subshrub. By applying the previous three steps for each of these branches, we first calculate the EAs of the upper extreme of the branch and finally the EAs of the representatives of the branches emerging from such an upper extreme. For example, if we start from the period-24 representative of the branch 1, we first calculate the EAs of the upper extreme of the branch 1, the five EAs of $M_{24,4}$, and then the representatives of the branches 11, 12, 13, and 14, and so on.

(b2) Second Subshrub, S_2

Let us calculate next the structure of the second subshrub. The second subshrub is constituted by what we call portions [23], each of which is born from one extreme of the first subshrub, $d_1 d_2 \cdots d_m 11 \cdots$. Let us focus on the portion S_2 that is born in $\overline{1}$, a Misiurewicz point $M_{16,1}$ that is a branch point of 4 branches (Figures 6(b) and 7). But $\overline{1}$ is both one starting point of S_2 and one end point of S_1. Therefore, the first and last EAs of $M_{16,1}$ can be calculated as an upper extreme of S_1. As we know from Section 2.3, in this case $(.\overline{b_1}, .\overline{b_2})$ corresponds to the first ancestor of $(.\overline{a_1}, .\overline{a_2})$, that is $(1/1) \cdot (q_1/p_1) \cdots (q_{N-i}/p_{N-i})$, where $N = 2$ and $i = 1$, or, what is the same, $1/4$ whose EAs are $(.\overline{b_1}, .\overline{b_2}) = (.\overline{0001}, .\overline{0010})$. To reach $\overline{1}$, we can start for example from the period-24 representative of the branch 1 and then $(.\overline{a_1}, .\overline{a_2}) = (.\overline{000100010001000100100010}, .\overline{000100010001001000010001})$. Hence, $PH^{(\infty)}[(.\overline{000100010001000100100010}, .\overline{000100010001001000010001}); (.\overline{01001}, .\overline{01010})] = (.0001000100010001\overline{0010}, .0001000100010010\overline{0001})$, which correspond to the first and last EAs of $M_{16,1}$.

As seen in Section 3.1, by using Properties 1 and 2 we calculate the two intermediate EAs of $\overline{1}$ and we obtain that the four EAs of this point, arranged from the lowest to highest, are $.0001000100010001\overline{0010}$, $.0001000100010001\overline{0100}$, $.0001000100010001\overline{1000}$, and $.0001000100010010\overline{0001}$. As seen in Section 3.2, by applying Property 3 we can calculate the EAs of the representatives of branches 1, 2, and 3 of periods 17, 18, and 19, respectively. Indeed, taking into account the third and fourth EAs, we calculate the period-17 representative of the branch 1, $(.\overline{a'_1}, .\overline{a'_2})_{b1} = (.\overline{00010001000100011}, .\overline{00010001000100100})$; taking into account the second and third EAs, we calculate the period-18 representative of the branch 2, $(.\overline{a'_1}, .\overline{a'_2})_{b2} = (.\overline{000100010001000101}, .\overline{000100010001000110})$; taking into account the first and second EAs, we calculate the period-19 representative of the branch 3, $(.\overline{a'_1}, .\overline{a'_2})_{b3} = (.\overline{0001000100010001001}, .\overline{0001000100010001010})$.

By applying the previous three steps to each of these branches, we firstly calculate the EAs of the upper extreme of the branch and finally the EAs of the representatives of the branches emerging from such an upper extreme. For example, if we start from the period-17 representative of the branch 1, we first calculate the EAs of upper extreme of the branch 1 and then the representatives of branches 11, 12, and 13, and so on. Obviously, it would be exactly the same for the other portions.

Let us note that in the first of the three steps, if we are in $\overline{1}$, $(.\overline{b_1}, .\overline{b_2})$ has to be the EAs of the first ancestor of $(.\overline{a_1}, .\overline{a_2})$ since $\overline{1}$ is an extreme of S_1; that is, the EAs of $(1/1) \cdot (q_1/p_1) \cdots (q_{N-i}/p_{N-i})$, where $N = 2$ and $i = 1$, or, what is the same, $1/4$ whose EAs are $(.\overline{b_1}, .\overline{b_2}) = (.\overline{0001}, .\overline{0010})$. For the following cases, $(.\overline{b_1}, .\overline{b_2})$ has to be the EAs of the second ancestor of $(.\overline{a_1}, .\overline{a_2})$ since we are in an extreme of a branch; that is, the EAs of $(1/1) \cdot (q_1/p_1) \cdots (q_{N-i+1}/p_{N-i+1})$ where $N = 2$ and now $i = 2$ because now we are in S_2, and once again we have the EAs of $1/4$, $(.\overline{b_1}, .\overline{b_2}) = (.\overline{0001}, .\overline{0010})$.

We will proceed in the same way for a tertiary,..., N-ary shrub. However, we do not show any example of these cases because of the difficulty of using so long binary expansions and, mainly, because they do not give any new contribution.

To finish, note that in all the cases the binary expansions of the EAs obtained from this three-step algorithm have not any effect of truncation or other numerical errors.

4. Conclusions

In this paper we calculate the EAs of the structural HCs and structural Misiurewicz points in the Mandelbrot set. To do this, we first review the tools we are going to use in this paper, as the harmonics and pseudoharmonics. Likewise, we review the concept of structure, the structure of a shrub, and the ancestral route, all of them widely described and clarified.

We introduce two new general properties of the external arguments of the Misiurewicz points, in order to introduce a new method to calculate the intermediate external arguments.

We also introduce a new third property that allows us to calculate the EAs of the representatives of the branches if the EAs of the extreme structural nodes of these branches are known.

We give an algorithm in three steps which allows us to calculate the EAs of all the structural components and structural nodes of a shrub, that is, it allows us to calculate the structure of any shrub and hence the structure of the Mandelbrot set.

Acknowledgments

This work was partially supported by the Ministry of Science and Innovation, R + D + i National Plan of Spain, Project TEC2009-13 964-C04-02, and the Ministry of Industry, Tourism and Commerce in collaboration with Telefónica I + D, Project SEGUR@, Conference CENIT-20072004.

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